

# Estimates on fractional higher derivatives of weak solutions for the Navier-Stokes equations

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## Abstract

We study weak solutions of the 3D Navier-Stokes equations in whole space with  $L^2$  initial data. It will be proved that  $\nabla^\alpha u$  is locally integrable in space-time for any real  $\alpha$  such that  $1 < \alpha < 3$ , which says that almost third derivative is locally integrable. Up to now, only second derivative  $\nabla^2 u$  has been known to be locally integrable by standard parabolic regularization. We also present sharp estimates of those quantities in weak- $L_{loc}^{4/(\alpha+1)}$ . These estimates depend only on the  $L^2$  norm of initial data and integrating domains. Moreover, they are valid even for  $\alpha \geq 3$  as long as  $u$  is smooth. The proof uses a good approximation of Navier-Stokes and a blow-up technique, which let us to focusing on a local study. For the local study, we use De Giorgi method with a new pressure decomposition. To handle non-locality of the fractional Laplacian, we will adopt some properties of the Hardy space and Maximal functions.

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## 1 Introduction and main result

In this paper, any derivative signs ( $\nabla, \Delta, (-\Delta)^{\alpha/2}, D, \partial$  and etc) denote derivatives in only space variable  $x \in \mathbb{R}^3$  unless time variable  $t \in \mathbb{R}$  is clearly specified. We study the 3-D Navier-Stokes equations

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla P - \Delta u &= 0 \quad \text{and} \\ \operatorname{div} u &= 0, \quad t \in (0, \infty), \quad x \in \mathbb{R}^3 \end{aligned} \quad (1)$$

with  $L^2$  initial data

$$u_0 \in L^2(\mathbb{R}^3), \quad \operatorname{div} u_0 = 0. \quad (2)$$

Regularity of weak solutions for the 3D Navier-Stokes equations has long history. Leray [27] 1930s and Hopf [22] 1950s proved existence of a global-time

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weak solution for any given  $L^2$  initial data. Such Leray-Hopf weak solutions  $u$  lie in  $L^\infty(0, \infty; L^2(\mathbb{R}^3))$  and  $\nabla u$  do in  $L^2(0, \infty; L^2(\mathbb{R}^3))$  and satisfy the energy inequality:

$$\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\|\nabla u\|_{L^2(0,t;L^2(\mathbb{R}^3))}^2 \leq \|u_0\|_{L^2(\mathbb{R}^3)}^2 \quad \text{for a.e. } t < \infty.$$

Until now, regularity and uniqueness of such weak solutions are generally open.

Instead, many criteria which ensure regularity of weak solutions have been developed. Among them the most famous one is Ladyženskaja-Prodi-Serrin Criteria ([24],[30] and [36]), which says: if  $u \in L^p((0, T); L^q(\mathbb{R}^3))$  for some  $p$  and  $q$  satisfying  $\frac{2}{p} + \frac{3}{q} = 1$  and  $p < \infty$ , then it is regular. Recently, the limit case  $p = \infty$  was established in the paper of Escauriaza, Serëgin and Šverák [16]. We may impose similar conditions to derivatives of velocity, vorticity or pressure. (see Beale, Kato and Majda [1], Beirão da Veiga [2] and Berselli and Galdi [4]) Also, many other conditions exist (e.g. see Cheskidov and Shvydkoy [10], Chan [9] and [5]).

On the other hand, many efforts have been given to measuring the size of possible singular set. This approach has been initiated by Scheffer [33]. Then, Caffarelli, Kohn and Nirenberg [6] improved the result and showed that possible singular sets have zero Hausdorff measure of one dimension for certain class of weak solutions (suitable weak solutions) satisfying the following additional inequality

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u \frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \leq 0 \quad (3)$$

in the sense of distribution. There are many other proofs of this fact (e.g. see Lin [28], [42] and Wolf [43]). Similar criteria for interior points with other quantities can be found in many places (e.g. see Struwe [40], Gustafson, Kang and Tsai [21], Serëgin [35] and Chae, Kang and Lee [8]). Also, Robinson and Sadowski [31] and Kukavica [23] studied box-counting dimensions of singular sets.

In this paper, our main concern is about space-time  $L_{(t,x)}^p = L_t^p L_x^p$  estimates of higher derivatives for weak solutions assuming only  $L^2$  initial data.  $\nabla u \in L^2((0, \infty) \times \mathbb{R}^3)$  is obvious from the energy inequality, and simple interpolation gives  $u \in L^{10/3}$ . For second derivatives of weak solutions, from standard parabolic regularization theory (see Ladyženskaja, Solonnikov and Ural'ceva [25]), we know  $\nabla^2 u \in L^{5/4}$  by considering  $(u \cdot \nabla)u$  as a source term. With different ideas, Constantin [12] showed  $L^{\frac{4}{3}-\epsilon}$  for any small  $\epsilon > 0$  in periodic setting, and later Lions [29] improved it up to weak- $L^{\frac{4}{3}}$  (or  $L^{\frac{4}{3}, \infty}$ ) by assuming  $\nabla u_0$  lying in the space of all bounded measures in  $\mathbb{R}^3$ . They used natural structure of the equation with some interpolation technique. On the other hand, Foiaş, Guillopé and Temam [18] and Duff [15] obtained other kinds of estimates for higher derivatives of weak solutions while Giga and Sawada [19] and Dong

and Du [14] covered mild solutions. For asymptotic behavior, we refer Schonbek and Wiegner [34].

Recently in [41], it has been shown that, for any small  $\epsilon > 0$ , any integer  $d \geq 1$  and any smooth solution  $u$  on  $(0, T)$ , we have bounds of  $\nabla^d u$  in  $L_{loc}^{\frac{4}{d+1}-\epsilon}$ , which depend only on  $L^2$  norm of initial data once we fix  $\epsilon$ ,  $d$  and the domain of integration. It can be considered as a natural extension of the result of Constantin [12] for higher derivatives. But the idea is completely different in the sense that [41] used the Galilean invariance of transport part of the equation and the partial regularity criterion in the version of [42], which re-proved the famous result of Caffarelli, Kohn and Nirenberg [6] by using a parabolic version of the De Giorgi method [13]. It is noteworthy that this method gave full regularity to the critical Surface Quasi-Geostrophic equation in [7]. The limit non-linear scaling  $p = \frac{4}{d+1}$  appears from the following invariance of the Navier-Stokes scaling  $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ :

$$\|\nabla^d u_\lambda\|_{L^p}^p = \lambda^{-1} \|\nabla^d u\|_{L^p}^p. \quad (4)$$

In this paper, our main result is better than the above result of [41] in the sense of the following three directions. First, we achieve the limit case weak- $L^{\frac{4}{d+1}}$  (or  $L^{\frac{4}{d+1}, \infty}$ ) as Lions [29] did for second derivatives. Second, we make similar bounds for fractional derivatives as well as classical derivatives. Last, we consider not only smooth solutions but also global-time weak solutions. These three improvements will give us that  $\nabla^{3-\epsilon} u$ , which is almost third derivatives of weak solutions, is locally integrable on  $(0, \infty) \times \mathbb{R}^3$ .

Our precise result is the following:

**Theorem 1.1.** *There exist universal constants  $C_{d,\alpha}$  which depend only on integer  $d \geq 1$  and real  $\alpha \in [0, 2)$  with the following two properties (I) and (II):*

(I) *Suppose that we have a smooth solution  $u$  of (1) on  $(0, T) \times \mathbb{R}^3$  for some  $0 < T \leq \infty$  with some initial data (2). Then it satisfies*

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u\|_{L^{p,\infty}(t_0, T; L^{p,\infty}(K))} \leq C_{d,\alpha} \left( \|u_0\|_{L^2(\mathbb{R}^3)}^2 + \frac{|K|}{t_0} \right)^{\frac{1}{p}} \quad (5)$$

*for any  $t_0 \in (0, T)$ , any integer  $d \geq 1$ , any  $\alpha \in [0, 2)$  and any bounded open subset  $K$  of  $\mathbb{R}^3$ , where  $p = \frac{4}{d+\alpha+1}$  and  $|\cdot|$  = the Lebesgue measure in  $\mathbb{R}^3$ .*

(II) *For any initial data (2), we can construct a suitable weak solution  $u$  of (1) on  $(0, \infty) \times \mathbb{R}^3$  such that  $(-\Delta)^{\frac{\alpha}{2}} \nabla^d u$  is locally integrable in  $(0, \infty) \times \mathbb{R}^3$  for  $d = 1, 2$  and for  $\alpha \in [0, 2)$  with  $(d + \alpha) < 3$ . Moreover, the estimate (5) holds with  $T = \infty$  under the same setting of the above part (I) as long as  $(d + \alpha) < 3$ .*

Let us begin with some simple remarks.

*Remark 1.1.* For any suitable weak solution  $u$ , we can define  $(-\Delta)^{\alpha/2}\nabla^d u$  in the sense of distributions  $\mathcal{D}'$  for any integer  $d \geq 0$  and for any real  $\alpha \in [0, 2)$ :

$$\langle (-\Delta)^{\alpha/2}\nabla^d u; \psi \rangle_{\mathcal{D}', \mathcal{D}} = (-1)^d \int_{(0, \infty) \times \mathbb{R}^3} u \cdot (-\Delta)^{\alpha/2}\nabla^d \psi \, dx dt \quad (6)$$

for any  $\psi \in \mathcal{D} = C_c^\infty((0, \infty) \times \mathbb{R}^3)$  where  $(-\Delta)^{\alpha/2}$  in the right hand side is the traditional fractional Laplacian in  $\mathbb{R}^3$  defined by the Fourier transform. Note that  $(-\Delta)^{\alpha/2}\nabla^d \psi$  lies in  $L_t^\infty L_x^2$ . Thus, this definition from (6) makes sense due to  $u \in L_t^\infty L_x^2$ . Note also  $(-\Delta)^0 = Id$ . For more general extensions of this fractional Laplacian operator, we recommend Silvestre [37].

*Remark 1.2.* Since we impose only (2) to  $u_0$ , the estimate (5) is a (quantitative) regularization result to higher derivatives. Also, in the proof, we will see that  $\|u_0\|_{L^2(\mathbb{R}^3)}^2$  in (5) can be relaxed to  $\|\nabla u\|_{L^2((0, T) \times \mathbb{R}^3)}^2$ . Thus it says that any (higher) derivatives can be controlled by having only  $L^2$  estimate of dissipation of energy.

*Remark 1.3.* The result of the part (I) for  $\alpha = 0$  extends the result of the previous paper [41] because for any  $0 < q < p < \infty$  and any bounded subset  $\Omega \subset \mathbb{R}^n$ , we have

$$\|f\|_{L^q(\Omega)} \leq C \cdot \|f\|_{L^{p, \infty}(\Omega)}$$

where  $C$  depends only on  $p, q$ , dimension  $n$  and Lebesgue measure of  $\Omega$  (e.g. see Grafakos [20]).

*Remark 1.4.* The assumption “smoothness” in the part (I) is about pure differentiability. For example, the result of the part (I) for  $d \geq 1$  and  $\alpha = 0$  holds once we know that  $u$  is  $d$ -times differentiable. In addition, constants in (5) are independent of any possible blow-time  $T$ .

*Remark 1.5.*  $p = 4/(d + \alpha + 1)$  is a very interesting relation as mentioned before. Due to this  $p$ , the estimate (5) is a non-linear estimate while many other *a priori* estimates are linear. Also, from the part (II) when  $(d + \alpha)$  is very close to 3, we can see that almost third derivatives of weak solutions are locally integrable. Moreover, imagine that the part (II) for  $d = \alpha = 0$  be true even though we can NOT prove it here. This would imply that this weak solution  $u$  could lie in  $L^{4, \infty}$  which is beyond the best known estimate  $u \in L^{10/3}$  from  $L^2$  initial data.

Before presenting the main ideas, we want to mention that Caffarelli, Kohn and Nirenberg [6] contains two different kinds of local regularity criteria. The first one is quantitative, and it says that if  $\|u\|_{L^3(Q(1))}$  and  $\|P\|_{L^{3/2}(Q(1))}$  is small, then  $u$  is bounded by some universal constant in  $Q(1/2)$ . The second one says that  $u$  is locally bounded near the origin if  $\limsup_{r \rightarrow 0} r^{-1} \|\nabla u\|_{L^2(Q(r))}^2$  is small. So it is qualitative in the sense that the conclusion says not that  $u$  is bounded by a universal constant but that  $\sup |u|$  for some local neighborhood is not infinite.

On the other hand, there is a different quantitative local regularity criterion in [42], which showed that for any  $p > 1$ , there exists  $\epsilon_p$  such that

$$\text{if } \|u\|_{L_t^\infty L_x^2(Q_1)} + \|\nabla u\|_{L_t^2 L_x^2(Q_1)} + \|P\|_{L_t^p L_x^1(Q_1)} \leq \epsilon_p, \text{ then } |u| \leq 1 \text{ in } Q_{1/2} \quad (7)$$

Recently, this criterion was used in [41] in order to obtain higher derivative estimates. The main proposition in [41] says that if both  $\| |\nabla u|^2 + |\nabla^2 P| \|_{L^1(Q(1))}$  and some other quantity about pressure are small, then  $u$  is bounded by 1 at the origin once  $u$  has a mean zero property in space. We can observe that  $\|\nabla u\|_{L^2(Q(1))}^2$  and  $\|\nabla^2 P\|_{L^1(Q(1))}$  have the same best scaling like (4) among all the other quantities which we can obtain from  $L^2$  initial data. However, the other quantity about pressure has a slightly worse scaling. That is the reason that the limit case  $L^{\frac{4}{\alpha+1}, \infty}$  has been missing in [41].

Here are the main ideas of proof. First, in order to obtain the missing limit case  $L^{\frac{4}{\alpha+1}, \infty}$ , we will see that it requires an equivalent estimate of (7) for  $p = 1$ . Here we extend this result up to  $p = 1$  for some approximation of the Navier-Stokes (see the proposition 2.1). To obtain this first goal, we will introduce a new pressure decomposition (see the lemma 3.3), which will be used in the De Giorgi-type argument. This makes us to remove the bad scaling term about pressure in [41]. As a result, by using the Galilean invariance property and some blow-up technique with the standard Navier-Stokes scaling, we can proceed our local study in order to obtain a better version of a quantitative partial regularity criterion for some approximation of the Navier-Stokes (see the proposition 2.2). As a result, we can prove  $L^{\frac{4}{\alpha+1}, \infty}$  estimate for classical derivatives ( $\alpha = 0$  case).

Second, the result for fractional derivatives ( $0 < \alpha < 2$  case) is not obvious at all because there is no proper interpolation theorem for  $L_{loc}^{p, \infty}$  spaces. For example, due to the non-locality of the fractional Laplacian operator, the fact  $\nabla^2 u \in L_{loc}^{\frac{4}{3}, \infty}$  with  $\nabla^3 u \in L_{loc}^{1, \infty}$  does not imply the case of fractional derivatives even if we assume  $u$  is smooth. Moreover, even though we assume that  $\nabla^2 u \in L^{\frac{4}{3}}(\mathbb{R}^3)$  and  $\nabla^3 u \in L^1(\mathbb{R}^3)$  which we can NOT prove here, the standard interpolation theorem still requires  $L^p(\mathbb{R}^3)$  for some  $p > 1$  (we refer Bergh and L fstr m [3]). To overcome the difficulty, we will use the Maximal functions of  $u$  which capture its behavior of long-range part. Unfortunately, second derivatives of pressure, which lie in the Hardy space  $\mathcal{H} \subset L^1(\mathbb{R}^3)$  from Coifman, Lions, Meyer and Semmes [11], do not have an integrable Maximal function since the Maximal operator is not bounded on  $L^1$ . In order to handle non-local parts of pressure, we will use some property of Hardy space, which says that some integrable functions play a similar role of the Maximal function (see (10)).

Finally, the result (II) for weak solutions comes from specific approximation of Navier-Stokes equations that Leray [27] used in order to construct a global time weak solution :  $\partial_t u_n + ((u_n * \phi_{(1/n)}) \cdot \nabla) u_n + \nabla P_n - \Delta u_n = 0$  and  $\text{div } u_n = 0$  where  $\phi$  is a fixed mollifier in  $\mathbb{R}^3$ , and  $\phi_{(1/n)}$  is defined by  $\phi_{(1/n)}(\cdot) = n^3 \phi(n \cdot)$ .

Main advantage for us of adopting this approximation is that it has strong existence theory of global-time smooth solutions  $u_n$  for each  $n$ , and it is well-known that there exists a suitable weak solution  $u$  as a weak limit. In fact, for any integer  $d \geq 1$  and for any  $\alpha \in [0, 2)$ , we will obtain bounds for  $u_n$  in the form of (5) with  $T = \infty$ , which is uniform in  $n$ . Since  $p = 4/(d + \alpha + 1)$  is greater than 1 for the case  $(d + \alpha) < 3$ , we can know that  $(-\Delta)^{\frac{\alpha}{2}} \nabla^d u$  exists as a locally integrable function from weak-compactness of  $L^p$  for  $p > 1$ .

However, to prove (5) uniformly for the approximation is nontrivial because our proof is based on local study while the approximation is not scaling-invariant with the standard Navier-Stokes scaling: After the scaling, the advection velocity  $u * \phi_{(1/n)}$  depends the original velocity  $u$  more non-locally than before. Moreover, when we consider the case of fractional derivatives of weak solutions, it requires even Maximal of Maximal functions to handle non-local parts of the advection velocity which depends the original velocity non-locally.

The paper is organized as follows. In the next section, preliminaries with the main propositions 2.1 and 2.2 will be introduced. Then we prove those propositions 2.1 and 2.2 in sections 3 and 4, respectively. Finally we will explain how the proposition 2.2 implies the part (II) of the theorem 1.1 for  $\alpha = 0$  and for  $0 < \alpha < 2$  in subsections 5.2 and 5.3 respectively while the part (I) will be covered in the subsection 5.4. After that, the appendix contains some missing proofs of technical lemmas.

## 2 Preliminaries, definitions and main propositions

We begin this section by fixing some notations and reminding some well-known results on analysis. After that we will present definitions of two approximations and two main propositions. In this paper, any derivatives, convolutions and Maximal functions are with respect to space variable  $x \in \mathbb{R}^3$  unless time variable is specified.

### Notations for general purpose

We define  $B(r) =$  the ball in  $\mathbb{R}^3$  centered at the origin with radius  $r$ ,  $Q(r) = (-r^2, 0) \times B(r)$ , the cylinder in  $\mathbb{R} \times \mathbb{R}^3$  and  $B(x; r) =$  the ball in  $\mathbb{R}^3$  centered at  $x$  with radius  $r$ .

To the end of this paper, we fix  $\phi \in C^\infty(\mathbb{R}^3)$  satisfying:

$$\int_{\mathbb{R}^3} \phi(x) dx = 1, \quad \text{supp}(\phi) \subset B(1), \quad 0 \leq \phi \leq 1$$

$$\phi(x) = 1 \text{ for } |x| \leq \frac{1}{2} \quad \text{and} \quad \phi \text{ is radial.}$$

For real number  $r > 0$ , we define functions  $\phi_r \in C^\infty(\mathbb{R}^3)$  by  $\phi_r(x) = \frac{1}{r^3} \phi(\frac{x}{r})$ . Moreover, for  $r = 0$ , we define  $\phi_r = \phi_0 = \delta_0$  as the Dirac-delta function, which implies that the convolution between  $\phi_0$  and any function becomes the function itself. From the Young's inequality for convolutions, we can observe

$$\|f * \phi_r\|_{L^p(B(a))} \leq \|f\|_{L^p(B(a+r))} \quad (8)$$

due to  $\text{supp}(\phi_r) \subset B(r)$  for any  $p \in [1, \infty]$ , for any  $f \in L^p_{loc}$  and for any  $a, r > 0$ .

### $L^p$ , weak- $L^p$ and Sobolev spaces $W^{n,p}$

Let  $K$  be a open subset  $K$  of  $\mathbb{R}^n$ . For  $0 < p < \infty$ , we define  $L^p(K)$  by the standard way with (quasi) norm  $\|f\|_{L^p(K)} = (\int_K |f|^p dx)^{1/p}$ . From the Banach-Alaoglu theorem, any sequence which is bounded in  $L^p(K)$  for  $p \in (1, \infty)$  has a weak limit from some subsequence due to the weak-compactness.

Also, for  $0 < p < \infty$ , the weak- $L^p(K)$  space (or  $L^{p,\infty}(K)$ ) is defined by

$$L^{p,\infty}(K) = \{f \text{ measurable in } K \subset \mathbb{R}^d : \sup_{\alpha > 0} \left( \alpha^p \cdot |\{|f| > \alpha\} \cap K| \right) < \infty\}$$

with (quasi) norm  $\|f\|_{L^{p,\infty}(K)} = \sup_{\alpha > 0} \left( \alpha \cdot |\{|f| > \alpha\} \cap K|^{1/p} \right)$ . From the Chebyshev's inequality, we have  $\|f\|_{L^{p,\infty}(K)} \leq \|f\|_{L^p(K)}$  for any  $0 < p < \infty$ . Also, for  $0 < q < p < \infty$ ,  $L^{p,\infty}(K) \subset L^q(K)$  once  $K$  is bounded (refer the remark 1.3 in the beginning).

For any integer  $n \geq 0$  and for any  $p \in [1, \infty]$ , we denote  $W^{n,p}(\mathbb{R}^3)$  and  $W^{n,p}(B(r))$  as the standard Sobolev spaces for the whole space  $\mathbb{R}^3$  and for any ball  $B(r)$  in  $\mathbb{R}^3$ , respectively.

### The Maximal function $\mathcal{M}$ and the Riesz transform $\mathcal{R}_j$

The Maximal function  $\mathcal{M}$  in  $\mathbb{R}^d$  is defined by the following standard way:

$$\mathcal{M}(f)(x) = \sup_{r > 0} \frac{1}{|B(r)|} \int_{B(r)} |f(x+y)| dy.$$

Also, we can express this Maximal operator as a supremum of convolutions:  $\mathcal{M}(f) = C \sup_{\delta > 0} \left( \chi_\delta * |f| \right)$  where  $\chi = \mathbf{1}_{\{|x| < 1\}}$  is the characteristic function of the unit ball, and  $\chi_\delta(\cdot) = (1/\delta^3) \chi(\cdot/\delta)$ . One of properties of the Maximal function is that  $\mathcal{M}$  is bounded from  $L^p(\mathbb{R}^d)$  to  $L^p(\mathbb{R}^d)$  for  $p \in (1, \infty]$  and from  $L^1(\mathbb{R}^d)$  to  $L^{1,\infty}(\mathbb{R}^d)$ . In this paper, we denote  $\mathcal{M}$  and  $\mathcal{M}^{(t)}$  as the Maximal functions in  $\mathbb{R}^3$  and in  $\mathbb{R}^1$ , respectively.

For  $1 \leq j \leq 3$ , the Riesz Transform  $\mathcal{R}_j$  in  $\mathbb{R}^3$  is defined by:

$$\widehat{\mathcal{R}_j(f)}(x) = i \frac{x_j}{|x|} \hat{f}(x)$$

for any  $f \in \mathcal{S}$  (the Schwartz space). Moreover we can extend such definition for functions  $L^p(\mathbb{R}^3)$  for  $1 < p < \infty$  and it is well-known that  $\mathcal{R}_j$  is bounded in  $L^p$  for the same range of  $p$ .

### The Hardy space $\mathcal{H}$

The Hardy space  $\mathcal{H}$  in  $\mathbb{R}^3$  is defined by

$$\mathcal{H}(\mathbb{R}^3) = \{f \in L^1(\mathbb{R}^3) \quad : \quad \sup_{\delta > 0} |\mathcal{P}_\delta * f| \in L^1(\mathbb{R}^3)\}$$

where  $\mathcal{P} = C(1 + |x|^2)^{-2}$  is the Poisson kernel and  $\mathcal{P}_\delta$  is defined by  $\mathcal{P}_\delta(\cdot) = \delta^{-3} \mathcal{P}(\cdot/\delta)$ . A norm of  $\mathcal{H}$  is defined by  $L^1$  norm of  $\sup_{\delta > 0} |\mathcal{P}_\delta * f|$ . Thus  $\mathcal{H}$  is a subspace of  $L^1(\mathbb{R}^3)$  and  $\|f\|_{L^1(\mathbb{R}^3)} \leq \|f\|_{\mathcal{H}(\mathbb{R}^3)}$  for any  $f \in \mathcal{H}$ . Moreover, the Riesz Transform is bounded from  $\mathcal{H}$  to  $\mathcal{H}$ .

One of important applications of the Hardy space is the compensated compactness (see Coifman, Lions, Meyer and Semmes [11]). Especially, it says that if  $E, B \in L^2(\mathbb{R}^3)$  and  $\operatorname{curl} E = \operatorname{div} B = 0$  in distribution, then  $E \cdot B \in \mathcal{H}(\mathbb{R}^3)$  and we have

$$\|E \cdot B\|_{\mathcal{H}(\mathbb{R}^3)} \leq C \cdot \|E\|_{L^2(\mathbb{R}^3)} \cdot \|B\|_{L^2(\mathbb{R}^3)}$$

for some universal constant  $C$ . In order to obtain some regularity of second derivative of pressure, we can combine compensated compactness with boundedness of the Riesz transform in  $\mathcal{H}(\mathbb{R}^3)$ . For example, if  $u$  is a weak solution of the Navier-Stokes (1), then a corresponding pressure  $P$  satisfies

$$\|\nabla^2 P\|_{L^1(0, \infty; \mathcal{H}(\mathbb{R}^3))} \leq C \cdot \|\nabla u\|_{L^2(0, \infty; L^2(\mathbb{R}^3))}^2 \quad (9)$$

(see Lions [29] or the lemma 7 in [41]).

Now it is well known that if we replace the Poisson kernel  $\mathcal{P}$  with any function  $\mathcal{G} \in C^\infty(\mathbb{R}^3)$  with compact support, then we have a constant  $C$  depending only on  $\mathcal{G}$  such that

$$\|\sup_{\delta > 0} |\mathcal{G}_\delta * f|\|_{L^1(\mathbb{R}^3)} \leq C \|\sup_{\delta > 0} |\mathcal{P}_\delta * f|\|_{L^1(\mathbb{R}^3)} = C \|f\|_{\mathcal{H}(\mathbb{R}^3)} \quad (10)$$

where  $\mathcal{G}_\delta(\cdot) = \mathcal{G}(\cdot/\delta)/\delta^3$ . (see Fefferman and Stein [17] or see Stein [39], Grafakos [20] for modern texts). Due to the supremum and the convolution in (10), we can say that even though the Maximal function  $\sup_{\delta > 0} (\chi_\delta * |f|)$  of any non-trivial Hardy space function  $f$  is not integrable, there exist at least integrable functions  $(\sup_{\delta > 0} |\mathcal{G}_\delta * f|)$ , which can capture non-local data as



Maximal functions do. However, note the position of the absolute value sign in (10), which is outside of the convolution while it is inside of the convolution for the Maximal function. It implies that (10) is slightly weaker than the Maximal function in the sense of controlling non-local data. This weakness is the reason that we introduce certain definitions of  $\zeta$  and  $h^\alpha$  in the following.

**Some notations which will be useful for fractional derivatives  $(-\Delta)^{\alpha/2}$**

The following two definitions of  $\zeta$  and  $h^\alpha$  will be used only in the proof for fractional derivatives. We define  $\zeta$  by  $\zeta(x) = \phi(\frac{x}{2}) - \phi(x)$ . Then we have

$$\begin{aligned} \zeta &\in C^\infty(\mathbb{R}^3), \quad \text{supp}(\zeta) \subset B(2), \quad \zeta(x) = 0 \text{ for } |x| \leq \frac{1}{2} \\ \text{and } \sum_{j=k}^{\infty} \zeta\left(\frac{x}{2^j}\right) &= 1 \text{ for } |x| \geq 2^k \text{ for any integer } k. \end{aligned} \quad (11)$$

In addition, we define function  $h^\alpha$  for  $\alpha > 0$  by  $h^\alpha(x) = \zeta(x)/|x|^{3+\alpha}$ . Also we define  $(h^\alpha)_\delta$  and  $(\nabla^d h^\alpha)_\delta$  by  $(h^\alpha)_\delta(x) = \delta^{-3} h^\alpha(x/\delta)$  and  $(\nabla^d h^\alpha)_\delta(x) = \delta^{-3} (\nabla^d h^\alpha)(x/\delta)$  for  $\delta > 0$  and for positive integer  $d$ , respectively. Then they satisfy

$$\begin{aligned} (h^\alpha)_\delta &\in C^\infty(\mathbb{R}^3), \quad \text{supp}((h^\alpha)_\delta) \subset B(2\delta) - B(\delta/2), \\ \text{and } \frac{1}{|x|^{3+\alpha}} \cdot \zeta\left(\frac{x}{2^j}\right) &= \frac{1}{(2^j)^\alpha} \cdot (h^\alpha)_{2^j}(x) \text{ for any integer } j. \end{aligned} \quad (12)$$

**The definition of the fractional Laplacian  $(-\Delta)^{\alpha/2}$**

For  $-3 < \alpha \leq 2$  and for  $f \in \mathcal{S}(\mathbb{R}^3)$  (the Schwartz space),  $(-\Delta)^{\frac{\alpha}{2}} f$  is defined by the Fourier transform:

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^\alpha \hat{f}(\xi) \quad (13)$$

Note that  $(-\Delta)^0 = Id$ . Especially, for  $\alpha \in (0, 2)$ , the fractional Laplacian can also be defined by the singular integral for any  $f \in \mathcal{S}$ :

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_\alpha \cdot P.V. \int_{\mathbb{R}^3} \frac{f(x) - f(y)}{|x - y|^{3+\alpha}} dy. \quad (14)$$

We introduce two notions of approximations to Navier-Stokes. The first one or (Problem I-n) is the approximation Leray [27] used while the second one or (Problem II-r) will be used in local study after we apply some certain scaling to (Problem I-n).

**Definition of (Problem I-n): the first approximation to Navier-Stokes**

**Definition 2.1.** Let  $n \geq 1$  be either an integer or the infinity  $\infty$ , and let  $0 < T \leq \infty$ . Suppose that  $u_0$  satisfy (2). We say that  $(u, P) \in [C^\infty((0, T) \times \mathbb{R}^3)]^2$  is a solution of (Problem I-n) on  $(0, T)$  for the data  $u_0$  if it satisfies

$$\begin{aligned} \partial_t u + ((u * \phi_{\frac{1}{n}}) \cdot \nabla) u + \nabla P - \Delta u &= 0 \\ \operatorname{div} u &= 0 \quad t \in (0, T), \quad x \in \mathbb{R}^3 \end{aligned} \quad (15)$$

and

$$u(t) \rightarrow u_0 * \phi_{\frac{1}{n}} \text{ in } L^2\text{-sense as } t \rightarrow 0. \quad (16)$$

*Remark 2.1.* When  $n = \infty$ , (15) is the Navier-Stokes on  $(0, T) \times \mathbb{R}^3$  with initial value  $u_0$ .

*Remark 2.2.* If  $n$  is not the infinity but an positive integer, then for any given  $u_0$  of (2), we have existence and uniqueness theory of (Problem I-n) on  $(0, \infty)$  with the energy equality

$$\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\|\nabla u\|_{L^2(0,t;L^2(\mathbb{R}^3))}^2 = \|u_0 * \phi_{\frac{1}{n}}\|_{L^2(\mathbb{R}^3)}^2. \quad (17)$$

for any  $t < \infty$  and it is well-known that we can extract a sub-sequence which converges to a suitable weak solution  $u$  of (1) and (3) with the initial data  $u_0$  of (2) by limiting procedure on a sequence of solutions of (Problem I-n) (see Leray [27], or see Lions [29], Lemarié-Rieusset [26] for modern texts).

*Remark 2.3.* As mentioned in the introduction section, we can observe that this notion (Problem I-n) is not invariant under the standard Navier-Stokes scaling  $u(t, x) \rightarrow \epsilon u(\epsilon^2 t, \epsilon x)$  due to the advection velocity  $(u * \phi_{1/n})$  unless  $n$  is the infinity.

**Definition of (Problem II-r): the second approximation to Navier-Stokes**

**Definition 2.2.** Let  $0 \leq r < \infty$  be real. We say that  $(u, P) \in [C^\infty((-4, 0) \times \mathbb{R}^3)]^2$  is a solution of (Problem II-r) if it satisfies

$$\begin{aligned} \partial_t u + (w \cdot \nabla) u + \nabla P - \Delta u &= 0 \\ \operatorname{div} u &= 0, \quad t \in (-4, 0), \quad x \in \mathbb{R}^3 \end{aligned} \quad (18)$$

where  $w$  is the difference of two functions:

$$w(t, x) = w'(t, x) - w''(t), \quad t \in (-4, 0), x \in \mathbb{R}^3 \quad (19)$$

which are defined by  $u$  in the following way:

$$w'(t, x) = (u * \phi_r)(t, x) \quad \text{and} \quad w''(t) = \int_{\mathbb{R}^3} \phi(y)(u * \phi_r)(t, y) dy.$$

*Remark 2.4.* This notion of (Problem II-r) gives us the mean zero property for the advection velocity  $w$ :  $\int_{\mathbb{R}^3} \phi(x)w(t, x)dx = 0$  on  $(-4, 0)$ . Also this  $w$  is divergent free from the definition. Moreover, by multiplying  $u$  to (18), we have

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(w \frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} = 0 \quad (20)$$

in classical sense because our definition needs  $u$  to be  $C^\infty$ .

*Remark 2.5.* We will introduce some specially designed  $\epsilon$ -scaling which is a bridge between (Problem I-n) and (Problem II-r) (it can be found in (96)). This scaling is based on the Galilean invariance in order to obtain the mean zero property for the velocity  $u$ :  $\int_{\mathbb{R}^3} \phi(x)u(t, x)dx = 0$  on  $(-4, 0)$ . Moreover, after this  $\epsilon$ -scaling is applied to solutions of (Problem I-n), the resulting functions will satisfy not conditions of (Problem II- $\frac{1}{n}$ ) but those of (Problem II- $\frac{1}{n\epsilon}$ ) (it can be found (97)). These things will be stated precisely in the section 5.

*Remark 2.6.* When  $r = 0$ , the equation (18) is the Navier-Stokes on  $(-4, 0) \times \mathbb{R}^3$  once we assume the mean zero property for  $u$ .

Now we present two main local-study propositions which require the notion of (Problem II-r). These are kinds of partial regularity theorems for solutions of (Problem II-r). The main difficulty to prove these two propositions is that  $\bar{\eta}$  and  $\bar{\delta} > 0$  should be independent of any  $r$  in  $[0, \infty)$ . We will prove this independence very carefully, which is the heart of the section 3 and 4.

### The first local study proposition for (Problem II-r)

The following one is a quantitative version of partial regularity theorems which extends that of [42] up to  $p = 1$ . The proof will be based on the De Giorgi iteration with a new pressure decomposition lemma 3.3 which will appear later.

**Proposition 2.1.** *There exists a  $\bar{\delta} > 0$  with the following property:*

*If  $u$  is a solution of (Problem II-r) for some  $0 \leq r < \infty$  verifying both*

$$\|u\|_{L^\infty(-2, 0; L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2, 0; L^1(B(1)))} + \|\nabla u\|_{L^2(-2, 0; L^2(B(\frac{5}{4})))} \leq \bar{\delta}$$

$$\text{and} \quad \|\mathcal{M}(|\nabla u|)\|_{L^2(-2, 0; L^2(B(2)))} \leq \bar{\delta},$$

*then we have*

$$|u(t, x)| \leq 1 \text{ on } [-\frac{3}{2}, 0] \times B(\frac{1}{2}).$$

The above proposition, whose proof will appear in the section 3, contains two bad scaling terms  $\|u\|_{L_t^\infty L_x^2}$  and  $\|P\|_{L_t^1 L_x^1}$ , while the following proposition 2.2 does not have those two. Instead, the proposition 2.2 will assume the mean-zero

property on  $u$  with the additional terms. We will see later that these additional ones have the best scaling like  $|\nabla u|^2$  (also, see (4)).

### The second local study proposition for (Problem II-r)

**Proposition 2.2.** *There exists a  $\bar{\eta} > 0$  and there exist constants  $C_{d,\alpha}$  depending only on  $d$  and  $\alpha$  with the following property:*

*If  $u$  is a solution of (Problem II-r) for some  $0 \leq r < \infty$  verifying both*

$$\int_{\mathbb{R}^3} \phi(x) u(t, x) dx = 0 \quad \text{for } t \in (-4, 0) \text{ and} \quad (21)$$

$$\int_{-4}^0 \int_{B(2)} \left( |\nabla u|^2(t, x) + |\nabla^2 P|(t, x) + |\mathcal{M}(|\nabla u|)|^2(t, x) \right) dx dt \leq \bar{\eta}, \quad (22)$$

*then  $|\nabla^d u| \leq C_{d,0}$  on  $Q(\frac{1}{3}) = (-\frac{1}{3})^2, 0) \times B(\frac{1}{3})$  for every integer  $d \geq 0$ .*

*Moreover if we assume further*

$$\begin{aligned} & \int_{-4}^0 \int_{B(2)} \left( |\mathcal{M}(\mathcal{M}(|\nabla u|))|^2 + |\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)|^{2/q} \right. \\ & \left. + |\mathcal{M}(|\nabla u|^q)|^{2/q} + \sum_{m=d}^{d+4} \sup_{\delta > 0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P|) \right) dx dt \leq \bar{\eta} \end{aligned} \quad (23)$$

*for some integer  $d \geq 1$  and for some real  $\alpha \in (0, 2)$  where  $q = 12/(\alpha + 6)$ , then  $|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u| \leq C_{d,\alpha}$  on  $Q(\frac{1}{6})$  for such  $(d, \alpha)$ .*

*Remark 2.7.* For the definitions of  $h^\alpha$  and  $(\nabla^{m-1} h^\alpha)_\delta$ , see around (12).

The proof will be given in the section 4 which will use the conclusion of the previous proposition 2.1. Moreover we will use an induction argument and the integral representation of the fractional Laplacian in order to get estimates for the fractional case. The Maximal function term of (22) is introduced to estimate non-local part of  $u$  while the Maximal of Maximal function terms of (23) is to estimate non-local part of  $w$  which is already non-local. On the other hand, because  $\nabla^2 P$  has only  $L^1$  integrability, we can not have  $L^1$  Maximal function of  $\nabla^2 P$ . Instead, we use some integrable functions, which is the last term of (23). This term plays the role which captures non-local information of pressure (see (10)). These will be stated clearly in sections 4 and 5.

## 3 Proof of the first local study proposition 2.1

This section is devoted to prove the proposition 2.1 which is a partial regularity theorem for (Problem II-r). Remember that we are looking for  $\bar{\delta}$  which

must be independent of  $r$ .

In the first subsection 3.1, we present some lemmas about the advection velocity  $w$  and a new pressure decomposition. After that, two big lemmas 3.4 and 3.5 in the subsections 3.2 and 3.3, which give us a control for big  $r$  and small  $r$  respectively, follow. Then the actual proof of the proposition 2.1 will appear in the last subsection 3.4 where we can combine those two big lemmas.

### 3.1 A control on the advection velocity $w$ and a new pressure decomposition

The following lemma says that convolution of any functions with  $\phi_r$  can be controlled by just one point value of the Maximal function with some factor of  $1/r$ . Of course, it is useful when  $r$  is away from 0.

**Lemma 3.1.** *Let  $f$  be an integrable function in  $\mathbb{R}^3$ . Then for any integer  $d \geq 0$ , there exists  $C = C(d)$  such that*

$$\|\nabla^d(f * \phi_r)\|_{L^\infty(B(2))} \leq \frac{C}{r^d} \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \inf_{x \in B(2)} \mathcal{M}f(x)$$

for any  $0 < r < \infty$ .

*Proof.* Let  $z, x \in B(2)$ . Then

$$\begin{aligned} |\nabla^d(f * \phi_r)(z)| &= |(f * \nabla^d \phi_r)(z)| = \left| \int_{B(z,r)} f(y) \nabla^d \phi_r(z-y) dy \right| \\ &\leq \|\nabla^d \phi_r\|_{L^\infty} \int_{B(z,r)} |f(y)| dy = \frac{\|\nabla^d \phi\|_{L^\infty}}{r^{d+3}} \int_{B(z,r)} |f(y)| dy \\ &\leq \frac{\|\nabla^d \phi\|_{L^\infty}}{r^{d+3}} \frac{(r+4)^3}{(r+4)^3} \int_{B(x,r+4)} |f(y)| dy \leq \frac{C}{r^d} \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \mathcal{M}f(x). \end{aligned}$$

We used  $B(z, r) \subset B(x, r+4)$ . Then we take sup in  $z$  and inf in  $x$ . Recall that  $\phi(\cdot)$  is the fixed function in this paper. □

The following corollary is just an application of the previous lemma to solutions of (Problem II-r).

**Corollary 3.2.** *Let  $u$  be a solution of (Problem II-r) for  $0 < r < \infty$ . Then for any integer  $d \geq 0$ , there exists  $C = C(d)$  such that*

$$\|w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))}$$

and

$$\|\nabla^d w\|_{L^2(-4,0;L^\infty(B(2)))} \leq \frac{C}{r^{d-1}} \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))}$$

if  $d \geq 1$ .

*Proof.* Recall  $\int_{\mathbb{R}^3} w(t, y)\phi(y)dy = 0$  and  $\text{supp}(\phi) \subset B(1)$ . Thus for  $z \in B(2)$

$$\begin{aligned} |w(t, z)| &= \left| \int_{\mathbb{R}^3} w(t, z)\phi(y)dy - \int_{\mathbb{R}^3} w(t, y)\phi(y)dy \right| \\ &\leq \|\nabla w(t, \cdot)\|_{L^\infty(B(2))} \int_{\mathbb{R}^3} |z - y|\phi(y)dy \\ &\leq C \|(\nabla u) * \phi_r(t, \cdot)\|_{L^\infty(B(2))} \cdot \int_{\mathbb{R}^3} \phi(y)dy \\ &\leq C \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \inf_{x \in B(2)} \mathcal{M}(|\nabla u|)(t, x). \end{aligned}$$

For last inequality, we used the lemma 3.1 to  $\nabla u$ . For  $d \geq 1$ , use  $\nabla^d w = \nabla^{d-1} [(\nabla u) * \phi_r]$ . □

To use De Giorgi type argument, we require more notations which will be used only in this section.

For real  $k \geq 0$ , define

$$\begin{aligned} B_k &= \text{the ball in } \mathbb{R}^3 \text{ centered at the origin with radius } \frac{1}{2}\left(1 + \frac{1}{2^{3k}}\right), \\ T_k &= -\frac{1}{2}\left(3 + \frac{1}{2^k}\right), \\ Q_k &= [T_k, 0] \times B_k \quad \text{and} \\ s_k &= \text{distance between } B_{k-1}^c \text{ and } B_{k-\frac{5}{6}} \\ &= 2^{-3k} \left( (\sqrt{2} - 1)2\sqrt{2} \right). \end{aligned} \tag{24}$$

Also we define  $s_\infty = 0$ . Note that  $0 < s_1 < \frac{1}{4}$  and the sequence  $\{s_k\}_{k=1}^\infty$  is strictly decreasing to zero as  $k$  goes to  $\infty$ .

For each integer  $k \geq 0$ , we define and fix a function  $\psi_k \in C^\infty(\mathbb{R}^3)$  satisfying:

$$\begin{aligned} \psi_k &= 1 \quad \text{in } B_{k-\frac{2}{3}}, \quad \psi_k = 0 \quad \text{in } B_{k-\frac{5}{6}}^c \\ 0 &\leq \psi_k(x) \leq 1, \quad |\nabla \psi_k(x)| \leq C2^{3k} \text{ and } |\nabla^2 \psi_k(x)| \leq C2^{6k} \text{ for } x \in \mathbb{R}^3. \end{aligned} \tag{25}$$

This  $\psi_k$  plays role of a cut-off function for  $B_k$ .

To prove the proposition 2.1, We need the following important lemma about pressure decomposition. Here we decompose our pressure term into three parts: a non-local part which depends on  $k$ , a local part which depends on  $k$  and a non-local part which does not depend on  $k$  and will be absorbed into the velocity component later.

**Lemma 3.3.** *There exists a constant  $\Lambda_1 > 0$  with the following property: Suppose  $A_{ij} \in L^1(B_0)$   $1 \leq i, j \leq 3$  and  $P \in L^1(B_0)$  with  $-\Delta P = \sum_{ij} \partial_i \partial_j A_{ij}$  in  $B_0$ . Then, there exist a function  $P_3$  with  $P_3|_{B_{\frac{2}{3}}} \in L^\infty$  such that, for any  $k \geq 1$ , we can decompose  $P$  by*

$$P = P_{1,k} + P_{2,k} + P_3 \quad \text{in } B_{\frac{1}{3}}, \quad (26)$$

and they satisfy

$$\|\nabla P_{1,k}\|_{L^\infty(B_{k-\frac{1}{3}})} + \|P_{1,k}\|_{L^\infty(B_{k-\frac{1}{3}})} \leq \Lambda_1 2^{12k} \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}, \quad (27)$$

$$-\Delta P_{2,k} = \sum_{ij} \partial_i \partial_j (\psi_k A_{ij}) \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad (28)$$

$$\|\nabla P_3\|_{L^\infty(B_{\frac{2}{3}})} \leq \Lambda_1 (\|P\|_{L^1(B_{\frac{1}{6}})} + \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}). \quad (29)$$

Note that  $\Lambda_1$  is a totally independent constant.

*Proof.* The product rule and the hypothesis imply

$$\begin{aligned} -\Delta(\psi_1 P) &= -\psi_1 \Delta P - 2 \operatorname{div}((\nabla \psi_1)P) + P \Delta \psi_1 \\ &= \psi_1 \sum_{ij} \partial_i \partial_j A_{ij} - 2 \operatorname{div}((\nabla \psi_1)P) + P \Delta \psi_1 \\ &= -\Delta P_{1,k} - \Delta P_{2,k} - \Delta P_3 \end{aligned}$$

where  $P_{1,k}$ ,  $P_{2,k}$  and  $P_3$  are defined by

$$\begin{aligned} -\Delta P_{1,k} &= \sum_{ij} \partial_i \partial_j ((\psi_1 - \psi_k) A_{ij}) \\ -\Delta P_{2,k} &= \sum_{ij} \partial_i \partial_j (\psi_k A_{ij}) \\ -\Delta P_3 &= -\sum_{ij} \partial_j [(\partial_i \psi_1)(A_{ij})] - \sum_{ij} \partial_i [(\partial_j \psi_1)(A_{ij})] \\ &\quad + \sum_{ij} (\partial_i \partial_j \psi_1)(A_{ij}) - 2 \operatorname{div}((\nabla \psi_1)P) + P \Delta \psi_1. \end{aligned}$$

$P_{1,k}$  and  $P_3$  are defined by the representation formula  $(-\Delta)^{-1}(f) = \frac{1}{4\pi}(\frac{1}{|x|} * f)$  while  $P_{2,k}$  by the Riesz transforms.

Since  $\psi_1 = 1$  on  $B_{\frac{1}{3}}$ , we have  $\Delta P = \Delta(\psi_1 P)$  on  $B_{\frac{1}{3}}$ . Thus (26) holds.

By definition of  $P_{2,k}$ , (28) holds.

For (27) and (29), it follows the proof of the lemma 3 of [42] directly. For completeness, we present a proof here. Note that  $(\psi_1 - \psi_k)$  is supported in  $(B_{\frac{1}{6}} - B_{k-\frac{2}{3}})$  and  $\nabla\psi_1$  is supported in  $(B_{\frac{1}{6}} - B_{\frac{1}{3}})$ . Thus for  $x \in B_{k-\frac{1}{3}}$ ,

$$\begin{aligned} |P_{1,k}(x)| &= \left| \frac{1}{4\pi} \int_{(B_{\frac{1}{6}} - B_{k-\frac{2}{3}})} \frac{1}{|x-y|} \sum_{ij} (\partial_i \partial_j ((\psi_1 - \psi_k) A_{ij}))(y) dy \right| \\ &\leq \sup_{y \in B_{k-\frac{2}{3}}} (|\nabla_y^2 \frac{1}{|x-y|}|) \cdot \sum_{ij} \int_{B_{\frac{1}{6}}} |A_{ij}(x)| dy \\ &\leq C \cdot \sup_{y \in B_{k-\frac{2}{3}}} \left( \frac{1}{|x-y|^3} \right) \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})} \leq C_1 \cdot 2^{9k} \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}. \end{aligned}$$

We used integration by parts and facts  $|x-y| \geq 2^{-3k}$  and  $|(\psi_1 - \psi_k)| \leq 1$ .

In the same way, for  $x \in B_{k-\frac{1}{3}}$ ,

$$|\nabla P_{1,k}(x)| \leq C_2 \cdot 2^{12k} \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}.$$

For  $x \in B_{\frac{2}{3}}$ ,

$$\begin{aligned} |\nabla P_3(x)| &= \left| \frac{1}{4\pi} \int_{(B_{\frac{1}{6}} - B_{\frac{1}{3}})} (\nabla_y \frac{1}{|x-y|}) \left[ - \sum_{ij} \partial_j [(\partial_i \psi_1)(A_{ij})] - \sum_{ij} \partial_i [(\partial_j \psi_1)(A_{ij})] \right. \right. \\ &\quad \left. \left. + \sum_{ij} (\partial_i \partial_j \psi_1)(A_{ij}) - 2 \operatorname{div}((\nabla \psi_1)P) + P \Delta \psi_1 \right] dy \right| \\ &\leq C_3 \left( \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})} + \|P\|_{L^1(B_{\frac{1}{6}})} \right). \end{aligned}$$

These prove (27) and (29) and we keep the constant  $\Lambda_1 = \max(C_1, C_2, C_3)$  for future use.  $\square$

Before presenting De Giori arguments for large  $r$  and small  $r$ , we need more notations. In the following two main lemmas 3.4 and 3.5,  $P_3$  will be constructed from solutions  $(u, P)$  for (Problem II-r) by using the previous lemma 3.3 and it will be clearly shown that  $\nabla P_3$  has  $L_t^1 L_x^\infty$  bound. Thus we can define

$$\begin{aligned} E_k(t) &= \frac{1}{2} (1 - 2^{-k}) + \int_{-1}^t \|\nabla P_3(s, \cdot)\|_{L^\infty(B_{\frac{2}{3}})} ds, \\ &\text{for } t \in [-2, 0] \text{ and for } k \geq 0. \end{aligned} \tag{30}$$



Note that  $E_k$  depends on  $t$ . We also define followings like in [42]

$$\begin{aligned} v_k &= (|u| - E_k)_+, \\ d_k &= \sqrt{\frac{E_k \mathbf{1}_{\{|u| \geq E_k\}}}{|u|} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u|^2} \quad \text{and} \\ U_k &= \sup_{t \in [T_k, 0]} \left( \int_{B_k} |v_k|^2 dx \right) + \int \int_{Q_k} |d_k|^2 dx dt \\ &= \|v_k\|_{L^\infty(T_k, 0; L^2(B_k))}^2 + \|d_k\|_{L^2(Q_k)}^2. \end{aligned}$$

In this way,  $P_3$  will be absorbed into  $v_k$ , which is the key idea of proof of this proposition 2.1.

### 3.2 De Giorgi argument to get a control for large $r$

The following big lemma says that we can obtain a certain uniform non-linear estimate in the form of  $W_k \leq C^k \cdot W_{k-1}^\beta$  when  $r$  is large. Then an elementary lemma can give us the conclusion (we will see the lemma 3.6 later). On the other hand, for small  $r$ , we have the factor of  $(1/r^3)$  which blows up as  $r$  goes to zero. This weak point implies that we still need some extra work after this lemma. (it will be the next big lemma 3.5).

**Lemma 3.4.** *There exist universal constants  $\delta_1 > 0$  and  $\bar{C}_1 > 1$  such that if  $u$  is a solution of (Problem II- $r$ ) for some  $0 < r < \infty$  verifying both*

$$\begin{aligned} &\|u\|_{L^\infty(-2, 0; L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2, 0; L^1(B(1)))} + \|\nabla u\|_{L^2(-2, 0; L^2(B(\frac{5}{4})))} \leq \delta_1 \\ &\text{and } \|\mathcal{M}(|\nabla u|)\|_{L^2(-2, 0; L^2(B(2)))} \leq \delta_1, \end{aligned}$$

*then we have*

$$U_k \leq \begin{cases} (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r \geq s_1 \\ \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r < s_1. \end{cases}$$

*Remark 3.1.*  $s_1$  is a pre-fixed constant defined in (24) such that  $0 < s_1 < 1/4$ , and  $(\delta_1, \bar{C}_1)$  is independent of any  $0 < r < \infty$ . It will be clear that the exponent  $7/6$  is not optimal and we can make it close to  $(4/3)$  arbitrarily. However, any exponent bigger than 1 is enough for our study.

*Proof.* We assume  $\delta_1 < 1$ . First we claim that there exists a universal constant  $\Lambda_2 \geq 1$  such that

$$\| |w| \cdot |u| \|_{L^2(-2, 0; L^{3/2}(B_{\frac{1}{6}}))} \leq \Lambda_2 \cdot \delta_1 \quad \text{for any } 0 < r < \infty. \quad (31)$$

In order to prove the above claim (31), we need to separate it into a large  $r$  case and a small  $r$  case:

**(I)-large  $r$  case.** From the corollary 3.2 if  $r \geq s_1$ , then

$$\begin{aligned} \|w\|_{L^2(-4,0;L^\infty(B(2)))} &\leq C \cdot \left(1 + \frac{4}{s_1}\right)^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \\ &\leq C \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \leq C\delta_1. \end{aligned} \quad (32)$$

Likewise,

$$\|\nabla w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C\delta_1. \quad (33)$$

With Holder's inequality and  $B_{\frac{1}{6}} \subset B_0 = B(1) \subset B(\frac{5}{4}) \subset B(2)$ ,

$$\begin{aligned} \| |w| \cdot |u| \|_{L^2(-2,0;L^{3/2}(B_{\frac{1}{6}}))} &\leq C \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} \cdot \|w\|_{L^2(-4,0;L^\infty(B(2)))} \\ &\leq C \cdot \delta_1^2 \leq C_1 \cdot \delta_1. \end{aligned}$$

so we obtained (31) for  $r \geq s_1$ .

**(II)-small  $r$  case.** On the other hand, if  $r < s_1$ , then

$$\begin{aligned} \|w\|_{L^2(-4,0;L^\infty(B(2)))} &\leq C \cdot \left(1 + \frac{4}{r}\right)^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \\ &\leq C \frac{1}{r^3} \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \leq C \frac{1}{r^3} \delta_1 \end{aligned} \quad (34)$$

and

$$\|\nabla w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \frac{1}{r^3} \delta_1. \quad (35)$$

However, it is not enough to prove (31) because  $\frac{1}{r^3}$  factor blows up as  $r$  goes to zero. So, instead, we use the idea that  $w$  and  $u$  are similar if  $r$  is small:

$$\|u\|_{L^4(-2,0;L^3(B_0))} \leq C \left( \|u\|_{L^\infty(-2,0;L^2(B_0))} + \|\nabla u\|_{L^2(-2,0;L^2(B_0))} \right) \leq C\delta_1$$

and

$$\|w'\|_{L^4(-2,0;L^3(B_{\frac{1}{6}}))} = \|u * \phi_r\|_{L^4(-2,0;L^3(B_{\frac{1}{6}}))} \leq \|u\|_{L^4(-2,0;L^3(B_0))} \leq C\delta_1$$

because  $u * \phi_r$  in  $B_{\frac{1}{6}}$  depends only on  $u$  in  $B_0$ . (recall that  $r \leq s_1$  and  $s_1$  is the distance  $B_0^c$  and  $B_{\frac{1}{6}}$  and refer (8)). For  $w''$ ,

$$\begin{aligned} \|w''\|_{L^\infty(-2,0;L^\infty(B(2)))} &= \|w''\|_{L_t^\infty((-2,0))} \\ &= \left\| \int_{\mathbb{R}^3} \phi(y)(u * \phi_r)(y) dy \right\|_{L_t^\infty((-2,0))} \\ &\leq C \|u * \phi_r\|_{L_x^1(B(1))} \|L_t^\infty((-2,0)) \\ &\leq C \|u\|_{L_x^1(B(\frac{5}{4}))} \|L_t^\infty((-2,0)) \\ &\leq C \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} \\ &\leq C\delta_1 \end{aligned} \quad (36)$$

because  $w''$  is a constant in  $x$ ,  $\text{supp}(\phi) \subset B(1)$  and  $u * \phi_r$  in  $B(1)$  depends only on  $u$  in  $B(1 + s_1)$  which is a subset of  $B(\frac{5}{4})$ . As a result, we have

$$\begin{aligned} \| |w| \cdot |u| \|_{L^2(-2,0;L^{3/2}(B_{\frac{1}{6}}))} &\leq C \|u\|_{L^4(-2,0;L^3(B(1)))} \cdot \|w\|_{L^4(-2,0;L^3(B(\frac{1}{6})))} \\ &\leq C \delta_1 \cdot \| |w'| + |w''| \|_{L^4(-2,0;L^3(B(\frac{1}{6})))} \\ &\leq C \cdot \delta_1^2 \leq C_2 \cdot \delta_1 \end{aligned} \quad (37)$$

so that we obtained (31) for  $r \leq s_1$ .

Hence, taking

$$\Lambda_2 = \max(C_1, C_2, 1), \quad (38)$$

we have (31) and  $\Lambda_2$  is independent of  $0 < r < \infty$  as long as  $\delta_1 < 1$ . From now on, we assume  $\delta_1 < 1$  sufficiently small to be  $10 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \leq 1/2$  (Recall that  $\Lambda_1$  comes from the lemma 3.3).

Thanks to the lemma 3.3 and (31), by putting  $A_{ij} = w_i u_j$  we can decompose  $P$  by

$$P = P_{1,k} + P_{2,k} + P_3 \quad \text{in } [-2, 0] \times B_{\frac{1}{3}}$$

for each  $k \geq 1$  with following properties:

$$\begin{aligned} \| |\nabla P_{1,k}| + |P_{1,k}| \|_{L^2(-2,0;L^\infty(B_{k-\frac{1}{3}}))} &\leq \Lambda_1 2^{12k} \sum_{ij} \|w_i u_j\|_{L^2(-2,0;L^1(B_{\frac{1}{6}}))} \\ &\leq 9 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \cdot 2^{12k} \leq 2^{12k} \quad \text{for any } k \geq 1, \end{aligned} \quad (39)$$

$$-\Delta P_{2,k} = \sum_{ij} \partial_i \partial_j (\psi_k w_i u_j) \quad \text{in } [-2, 0] \times \mathbb{R}^3 \quad \text{for any } k \geq 1 \quad \text{and} \quad (40)$$

$$\begin{aligned} \|\nabla P_3\|_{L^1(-2,0;L^\infty(B_{\frac{2}{3}}))} &\leq \Lambda_1 \left( \|P\|_{L^1(-2,0;L^1(B(1)))} + \sum_{ij} \|w_i u_j\|_{L^2(-2,0;L^1(B(1)))} \right) \\ &\leq \Lambda_1 (\delta_1 + 9 \cdot \Lambda_2 \cdot \delta_1) \leq 10 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \leq \frac{1}{2}. \end{aligned} \quad (41)$$

Note that the above (41) enables  $E_k$  to be well-defined and it satisfies  $0 \leq E_k \leq 1$  (see the definition of  $E_k$  in (30)).

In the following remarks 3.2–3.4, we gather some easy results, which were obtained in [42], without proof. They can be found in the lemmas 4, 6 and the remark of the lemma 4 of [42]. Note that any constants  $C$  in the following remarks do not depend on  $k$ .

*Remark 3.2.* For any  $k \geq 0$ , the function  $u$  can be decomposed by  $u = u \frac{v_k}{|u|} + u(1 - \frac{v_k}{|u|})$ . Also we have

$$\begin{aligned} \left| u(1 - \frac{v_k}{|u|}) \right| &\leq 1, \quad \frac{v_k}{|u|} |\nabla u| \leq d_k, \quad \mathbf{1}_{|u| \geq E_k} |\nabla u| \leq d_k, \\ |\nabla v_k| &\leq d_k \quad \text{and} \quad \left| \nabla \frac{uv_k}{|u|} \right| \leq 3d_k. \end{aligned} \quad (42)$$

*Remark 3.3.* For any  $k \geq 1$  and for any  $q \geq 1$ ,

$$\|\mathbf{1}_{v_k > 0}\|_{L^q(Q_{k-1})} \leq C 2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}} \quad \text{and} \quad \|\mathbf{1}_{v_k > 0}\|_{L^\infty(T_{k-1,0}; L^q(Q_{k-1}))} \leq C 2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}.$$

*Remark 3.4.* For any  $k \geq 1$ ,  $\|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})} \leq C U_{k-1}^{\frac{1}{2}}$ .

From the above remarks 3.2–3.4, we have for any  $1 \leq p \leq \frac{10}{3}$ ,

$$\begin{aligned} \|v_k\|_{L^p(Q_{k-1})} &= \|v_k \mathbf{1}_{v_k > 0}\|_{L^p(Q_{k-1})} \\ &\leq \|v_k\|_{L^{\frac{10}{3}}(Q_{k-1})} \cdot \|\mathbf{1}_{v_k > 0}\|_{L^{1/(\frac{1}{p} - \frac{3}{10})}(Q_{k-1})} \\ &\leq \|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})} \cdot C 2^{\frac{10k}{3} \cdot (\frac{1}{p} - \frac{3}{10})} U_{k-1}^{\frac{5}{3} \cdot (\frac{1}{p} - \frac{3}{10})} \\ &\leq C 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}}. \end{aligned} \quad (43)$$

Likewise, for any  $1 \leq p \leq 2$ ,

$$\|v_k\|_{L^\infty(T_{k-1,0}; L^p(B_{k-1}))} \leq C 2^k U_{k-1}^{\frac{1}{p}} \quad (44)$$

and

$$\|d_k\|_{L^p(Q_{k-1})} \leq C 2^{\frac{5k}{3}} U_{k-1}^{\frac{5}{3p} - \frac{1}{3}}. \quad (45)$$

Second, we claim that for every  $k \geq 1$ , functions  $v_k$  verifies:

$$\begin{aligned} \partial_t \frac{v_k^2}{2} + \operatorname{div}(w \frac{v_k^2}{2}) + d_k^2 - \Delta \frac{v_k^2}{2} \\ + \operatorname{div}(u(P_{1,k} + P_{2,k})) + (\frac{v_k}{|u|} - 1)u \cdot \nabla(P_{1,k} + P_{2,k}) \leq 0 \end{aligned} \quad (46)$$

in  $(-2, 0) \times B_{\frac{2}{3}}$ .

*Remark 3.5.* Note that the above inequality (46) does not contain the  $P_3$  term. We will see that this fact comes from the definition of  $E_k(t)$  in (30).

Indeed, observe that  $\frac{v_k^2}{2} = \frac{|u|^2}{2} + \frac{v_k^2 - |u|^2}{2}$  and note that  $E_k$  does not depend on space variable but on time variable. Thus we can compute, for time derivatives,

$$\begin{aligned} \partial_t \left( \frac{v_k^2 - |u|^2}{2} \right) &= v_k \partial_t v_k - u \partial_t u = v_k \partial_t |u| - v_k \partial_t E_k - u \partial_t u \\ &= u \left( \frac{v_k}{|u|} - 1 \right) \partial_t u - v_k \partial_t E_k = u \left( \frac{v_k}{|u|} - 1 \right) \partial_t u - v_k \|\nabla P_3(t, \cdot)\|_{L^\infty(B_{\frac{2}{3}})} \end{aligned}$$

while, for any space derivatives  $\partial_\alpha$ ,

$$\partial_\alpha \left( \frac{v_k^2 - |u|^2}{2} \right) = u \left( \frac{v_k}{|u|} - 1 \right) \partial_\alpha u.$$

Then we follow the same way as the lemma 5 of [42] did: First, we multiply (18) by  $u(\frac{v_k}{|u|} - 1)$ , and then we sum the result and (20). We omit the details which can be found in the proof of the lemma 5 of [42]. As a result, we have

$$\begin{aligned} 0 &\geq \partial_t \frac{v_k^2}{2} + \operatorname{div}(w \frac{v_k^2}{2}) + d_k^2 - \Delta \frac{v_k^2}{2} + v_k \|\nabla P_3(t, \cdot)\|_{L^\infty(B_{\frac{2}{3}})} \\ &\quad + \operatorname{div}(uP) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P \\ &= \partial_t \frac{v_k^2}{2} + \operatorname{div}(w \frac{v_k^2}{2}) + d_k^2 - \Delta \frac{v_k^2}{2} + \left( v_k \|\nabla P_3(t, \cdot)\|_{L^\infty(B_{\frac{2}{3}})} + \frac{v_k}{|u|} u \cdot \nabla P_3 \right) \\ &\quad + \operatorname{div}(u(P_{1,k} + P_{2,k})) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla (P_{1,k} + P_{2,k}). \end{aligned}$$

For the last equality, we used the fact  $P = P_{1,k} + P_{2,k} + P_3$  in  $B_{\frac{1}{3}}$  and

$$\operatorname{div}(uP_3) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P_3 = \frac{v_k}{|u|} u \cdot \nabla P_3. \quad (47)$$

Thus we proved the claim (46) due to

$$v_k \|\nabla P_3(t, \cdot)\|_{L^\infty(B_{\frac{2}{3}})} + \frac{v_k}{|u|} u \cdot \nabla P_3 \geq 0 \quad \text{on } (-2, 0) \times B_{\frac{2}{3}}.$$

For any integer  $k$ , we introduce a cut-off function  $\eta_k(x) \in C^\infty(\mathbb{R}^3)$  satisfying

$$\begin{aligned} \eta_k &= 1 \quad \text{in } B_k, \quad \eta_k = 0 \quad \text{in } B_{k-\frac{1}{3}}^C, \quad 0 \leq \eta_k \leq 1, \\ |\nabla \eta_k| &\leq C2^{3k} \quad \text{and} \quad |\nabla^2 \eta_k| \leq C2^{6k}, \quad \text{for any } x \in \mathbb{R}^3. \end{aligned}$$

Multiplying (46) by  $\eta_k$  and integrating  $[\sigma, t] \times \mathbb{R}^3$  for  $T_{k-1} \leq \sigma \leq T_k \leq t \leq 0$ ,

$$\begin{aligned} &\int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t, x)|^2}{2} dx + \int_\sigma^t \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s, x) dx ds \\ &\leq \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(\sigma, x)|^2}{2} dx \\ &+ \int_\sigma^t \int_{\mathbb{R}^3} (\nabla \eta_k)(x) w(s, x) \frac{|v_k(s, x)|^2}{2} dx ds + \int_\sigma^t \int_{\mathbb{R}^3} (\Delta \eta_k)(x) \frac{|v_k(s, x)|^2}{2} dx ds \\ &- \int_\sigma^t \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u(P_{1,k} + P_{2,k})) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla (P_{1,k} + P_{2,k}) \right) (s, x) dx ds. \end{aligned}$$

Integrating in  $\sigma \in [T_{k-1}, T_k]$  and dividing by  $-(T_{k-1} - T_k) = 2^{-(k+1)}$ ,

$$\begin{aligned} & \sup_{t \in [T_k, 1]} \left( \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t, x)|^2}{2} dx + \int_{T_k}^t \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s, x) dx ds \right) \\ & \leq 2^{k+1} \cdot \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(\sigma, x)|^2}{2} dx \\ & + \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \nabla \eta_k(x) w(s, x) \frac{|v_k(s, x)|^2}{2} dx \right| ds + \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \Delta \eta_k(x) \frac{|v_k(s, x)|^2}{2} dx \right| ds \\ & + \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u(P_{1,k} + P_{2,k})) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla (P_{1,k} + P_{2,k}) \right) (s, x) dx \right| ds. \end{aligned}$$

From  $\eta_k = 1$  on  $B_k$ ,

$$\begin{aligned} U_k & \leq \sup_{t \in [T_k, 1]} \left( \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t, x)|^2}{2} dx \right) + \int_{T_k}^0 \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s, x) dx ds \\ & \leq 2 \cdot \sup_{t \in [T_k, 1]} \left( \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t, x)|^2}{2} dx + \int_{T_k}^t \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s, x) dx ds \right). \end{aligned}$$

Thus we have

$$U_k \leq (I) + (II) + (III) + (IV) \quad (48)$$

where

$$\begin{aligned} (I) & = C2^{6k} \int_{Q_{k-1}} |v_k(s, x)|^2 dx ds, \\ (II) & = \int_{Q_{k-1}} |\nabla \eta_k(x)| \cdot |w(s, x)| \cdot |v_k(s, x)|^2 dx ds, \\ (III) & = 2 \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u P_{1,k}) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P_{1,k} \right) (s, x) dx \right| ds \quad \text{and} \\ (IV) & = 2 \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u P_{2,k}) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P_{2,k} \right) (s, x) dx \right| ds. \end{aligned} \quad (49)$$

For (I), by using (43), for any  $0 < r < \infty$ ,

$$(I) = C2^{6k} \|v_k\|_{L^2(Q_{k-1})}^2 \leq C2^{10k} U_{k-1}^{\frac{5}{3}}. \quad (50)$$

For (II) with  $r \geq s_1$ , by using (32) and (44),

$$\begin{aligned}
(II) &\leq C2^{3k} \|w\|_{L^2(-4,0;L^\infty(B(2)))} \cdot \| |v_k|^2 \|_{L^2(T_{k-1},0;L^1(B_{k-1}))} \\
&\leq C2^{3k} \delta_1 \|v_k\|_{L^\infty(T_{k-1},0;L^{\frac{6}{5}}(B_{k-1}))} \cdot \|v_k\|_{L^2(T_{k-1},0;L^6(B_{k-1}))} \\
&\leq C2^{4k} \delta_1 U_{k-1}^{\frac{5}{6}} \cdot \left( \|v_{k-1}\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))} + \|\nabla v_{k-1}\|_{L^2(T_{k-1},0;L^2(B_{k-1}))} \right) \\
&\leq C2^{4k} \cdot \delta_1 \cdot U_{k-1}^{\frac{5}{6}} \cdot U_{k-1}^{\frac{1}{2}} \leq C2^{4k} \cdot \delta_1 \cdot U_{k-1}^{\frac{4}{3}} \leq C2^{4k} \cdot U_{k-1}^{\frac{4}{3}}.
\end{aligned} \tag{51}$$

For  $r < s_1$ , follow the above steps using (34) instead of using (32) then we get

$$(II) \leq C \frac{1}{r^3} 2^{4k} \cdot U_{k-1}^{\frac{4}{3}}. \tag{52}$$

For (III) (non-local pressure term), observe that

$$\operatorname{div}(uP_{1,k}) + \left(\frac{v_k}{|u|} - 1\right)u \cdot \nabla P_{1,k} = \frac{v_k}{|u|}u \cdot \nabla P_{1,k}$$

because everything is smooth. Thus, by using (39) and (43), for any  $0 < r < \infty$ ,

$$\begin{aligned}
(III) &\leq C \cdot \left\| \frac{v_k}{|u|}u \cdot \nabla P_{1,k} \right\|_{L^1(Q_{k-1})} \leq C \| |v_k| \cdot |\nabla P_{1,k}| \|_{L^1(Q_{k-1})} \\
&\leq \|v_k\|_{L^2(T_{k-1},0;L^1(B_{k-1}))} \cdot \|\nabla P_{1,k}\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \\
&\leq \|\mathbf{1}_{v_k > 0}\|_{L^2(T_{k-1},0;L^2(B_{k-1}))} \|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))} \cdot 2^{12k} \\
&\leq C2^{\frac{43k}{3}} U_{k-1}^{\frac{5}{6}} U_{k-1}^{\frac{1}{2}} \leq C2^{\frac{43k}{3}} U_{k-1}^{\frac{4}{3}}.
\end{aligned} \tag{53}$$

For (IV) (local pressure term), as we did for (III), observe

$$\operatorname{div}(uP_{2,k}) + \left(\frac{v_k}{|u|} - 1\right)u \cdot \nabla P_{2,k} = \frac{v_k}{|u|}u \cdot \nabla P_{2,k}.$$

By definition of  $P_{2,k}$ , we have

$$\begin{aligned}
-\Delta P_{2,k} &= \sum_{ij} \partial_i \partial_j (\psi_k w_i u_j) = \sum_{ij} \partial_i ((\partial_j \psi_k) w_i u_j + \psi_k (\partial_j w_i) u_j) \\
&= \sum_{ij} \partial_i \left( (\partial_j \psi_k) w_i u_j \left(1 - \frac{v_k}{|u|}\right) + (\partial_j \psi_k) w_i u_j \frac{v_k}{|u|} \right. \\
&\quad \left. + \psi_k (\partial_j w_i) u_j \left(1 - \frac{v_k}{|u|}\right) + \psi_k (\partial_j w_i) u_j \frac{v_k}{|u|} \right)
\end{aligned}$$

and

$$\begin{aligned}
-\Delta(\nabla P_{2,k}) &= \sum_{ij} \partial_i \nabla \left( (\partial_j \psi_k) w_i u_j \left(1 - \frac{v_k}{|u|}\right) + (\partial_j \psi_k) w_i u_j \frac{v_k}{|u|} \right. \\
&\quad \left. + \psi_k (\partial_j w_i) u_j \left(1 - \frac{v_k}{|u|}\right) + \psi_k (\partial_j w_i) u_j \frac{v_k}{|u|} \right).
\end{aligned}$$

Thus we can decompose  $\nabla P_{2,k}$  by the Riesz transform into

$$\nabla P_{2,k} = G_{1,k} + G_{2,k} + G_{3,k} + G_{4,k}$$

where

$$\begin{aligned} G_{1,k} &= \sum_{ij} (\partial_i \nabla) (-\Delta)^{-1} \left( (\partial_j \psi_k) w_i u_j \left(1 - \frac{v_k}{|u|}\right) \right), \\ G_{2,k} &= \sum_{ij} (\partial_i \nabla) (-\Delta)^{-1} \left( (\partial_j \psi_k) w_i u_j \frac{v_k}{|u|} \right), \\ G_{3,k} &= \sum_{ij} (\partial_i \nabla) (-\Delta)^{-1} \left( \psi_k (\partial_j w_i) u_j \left(1 - \frac{v_k}{|u|}\right) \right) \quad \text{and} \\ G_{4,k} &= \sum_{ij} (\partial_i \nabla) (-\Delta)^{-1} \left( \psi_k (\partial_j w_i) u_j \frac{v_k}{|u|} \right). \end{aligned}$$

From  $L^p$ -boundedness of the Riesz transform with the fact  $\text{supp}(\psi_k) \subset B_{k-(5/6)} \subset B_{k-1}$ , we have

$$\begin{aligned} \|G_{2,k}\|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} &\leq C 2^{3k} \|w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \cdot \|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))}, \\ \|G_{4,k}\|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} &\leq C \cdot \|\nabla w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \cdot \|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))}. \end{aligned}$$

For any  $1 < p < \infty$ ,

$$\begin{aligned} \|G_{1,k}\|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} &\leq C_p \cdot 2^{3k} \|w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \quad \text{and} \\ \|G_{3,k}\|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} &\leq C_p \cdot \|\nabla w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))}. \end{aligned}$$

Therefore, by using (33) and (35)

$$\| |G_{2,k}| + |G_{4,k}| \|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} \leq \begin{cases} C \cdot 2^{3k} \cdot U_{k-1}^{\frac{1}{2}}, & \text{if } r \geq s_1 \\ C \cdot 2^{3k} \cdot \frac{1}{r^3} \cdot U_{k-1}^{\frac{1}{2}}, & \text{if } r < s_1 \end{cases}$$

and, for any  $1 < p < \infty$ ,

$$\| |G_{1,k}| + |G_{3,k}| \|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} \leq \begin{cases} C_p \cdot 2^{3k}, & \text{if } r \geq s_1 \\ C_p \cdot 2^{3k} \cdot \frac{1}{r^3}, & \text{if } r < s_1. \end{cases}$$

Thus, by using the above estimates and (43), for  $r \geq s_1$  and  $p > 5$ ,

$$\begin{aligned} (IV) &\leq C \cdot \left\| \frac{v_k}{|u|} u \cdot \nabla P_{2,k} \right\|_{L^1(Q_{k-1})} \leq C \| |v_k| \cdot |\nabla P_{2,k}| \|_{L^1(Q_{k-1})} \\ &\leq C \| |v_k| \cdot (|G_{1,k}| + |G_{3,k}|) \|_{L^1(Q_{k-1})} + C \| |v_k| \cdot (|G_{2,k}| + |G_{4,k}|) \|_{L^1(Q_{k-1})} \\ &\leq \|v_k\|_{L^2(T_{k-1},0;L^{\frac{p}{p-1}}(B_{k-1}))} \cdot \| |G_{1,k}| + |G_{3,k}| \|_{L^2(T_{k-1},0;L^p(B_{k-1}))} \\ &\quad + \|v_k\|_{L^2(T_{k-1},0;L^2(B_{k-1}))} \cdot \| |G_{2,k}| + |G_{4,k}| \|_{L^2(T_{k-1},0;L^2(B_{k-1}))} \\ &\leq C \cdot C_p \cdot 2^{\frac{16k}{3}} U_{k-1}^{\frac{4p-5}{3p}}. \end{aligned}$$



By the same way, for  $r < s_1$  and  $p > 5$ ,

$$(IV) \leq C \cdot C_p \cdot \frac{1}{r^3} 2^{\frac{16k}{3}} U_{k-1}^{\frac{4p-5}{3p}}.$$

Thus, by taking  $p = 10$ ,

$$(IV) \leq \begin{cases} C \cdot 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}, & \text{if } r \geq s_1 \\ C \cdot \frac{1}{r^3} 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}, & \text{if } r < s_1. \end{cases} \quad (54)$$

Finally, combining (50), (51), (52), (53) and (54) gives us

$$(I) + (II) + (III) + (IV) \leq \begin{cases} C^k \cdot U_{k-1}^{\frac{7}{6}}, & \text{if } r \geq s_1 \\ \frac{1}{r^3} \cdot C^k \cdot U_{k-1}^{\frac{7}{6}}, & \text{if } r < s_1. \end{cases}$$

□

### 3.3 De Giorgi argument to get a control for small $r$

The following big lemma makes us be able to avoid the weak point of the previous lemma 3.4 when we handle small  $r$  including the case  $r = 0$ .

Recall the definition of  $s_k$  in (24) first. It is the distance between  $B_{k-1}^c$  and  $B_{k-\frac{5}{6}}$  and  $s_k$  is strictly decreasing to zero as  $k \rightarrow \infty$ . For any  $0 < r < s_1$  we define  $k_r$  as the integer such that  $s_{k_r+1} < r \leq s_{k_r}$ . Note that  $k_r$  is integer-valued,  $k_r \geq 1$  and is increasing to  $\infty$  as  $r$  goes to zero. For the case  $r = 0$ , we simply define  $k_r = k_0 = \infty$ .

**Lemma 3.5.** *There exist universal constants  $\delta_2$  and  $\bar{C}_2 > 1$  such that if  $u$  is a solution of (Problem II- $r$ ) for some  $0 \leq r < s_1$  verifying both*

$$\begin{aligned} & \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \leq \delta_2 \\ & \text{and } \|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^2(B(2)))} \leq \delta_2, \end{aligned}$$

then we have

$$U_k \leq (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} \quad \text{for any integer } k \text{ such that } 1 \leq k \leq k_r.$$

*Remark 3.6.* Note that  $\delta_2$  and  $\bar{C}_2$  are independent of any  $r \in [0, s_1)$  and the exponent  $7/6$  is not optimal and we can make it almost  $(4/3)$ .

*Remark 3.7.* This lemma says that even though  $r$  is very small, we can make the above uniform estimate for the first few steps  $k \leq k_r$ . Moreover, the number  $k_r$  of these steps is increasing to the infinity with a certain rate as  $r$  goes to zero. In the subsection 3.4, we will see that this rate is enough to obtain a uniform estimate for any small  $r$  once we combine two lemmas 3.4 and 3.5.

*Proof.* In this proof, we can borrow any inequalities in the proof of the previous lemma 3.4 except those which depend on  $r$  and blow up as  $r$  goes to zero.

Let  $0 \leq r < s_1$  and take any integer  $k$  such that  $1 \leq k \leq k_r$ . Like  $\delta_1$  of the previous lemma 3.4, we assume  $\delta_2$  so small that

$$\delta_2 < 1, \quad 10\Lambda_1\Lambda_2\delta_2 \leq \frac{1}{2}.$$

We begin this proof by decomposing  $w'$  by

$$w' = u * \phi_r = \left(u(1 - \frac{v_k}{|u|})\right) * \phi_r + \left(u \frac{v_k}{|u|}\right) * \phi_r = w'^1 + w'^2.$$

Thus the advection velocity  $w$  has a new decomposition:  $w = w' - w'' = (w'^1 + w'^2) - w'' = (w'^1 - w'') + w'^2$ . We will verify that  $w'^1 - w''$  is bounded and  $w'^2$  can be controlled locally. First, for  $w'^1$ ,

$$|w'^1(t, x)| = \left| \left( \left( u(1 - \frac{v_k}{|u|}) \right) * \phi_r \right)(t, x) \right| \leq \|u(1 - \frac{v_k}{|u|})(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq 1 \quad (55)$$

for any  $-4 \leq t$  and any  $x \in \mathbb{R}^3$ . From (36), we still have

$$\|w''\|_{L^\infty(-2, 0; L^\infty(B(2)))} \leq C\bar{\delta} \leq C. \quad (56)$$

Combining above two results,

$$\|w'^1\| + \|w''\|_{L^\infty(-2, 0; L^\infty(B(2)))} \leq C. \quad (57)$$

For  $w'^2$ , we observe that any  $L^p$  norm of  $w'^2 = \left(u \frac{v_k}{|u|}\right) * \phi_r$  in  $B_{k-\frac{5}{6}}$  is less than or equal to that of  $v_k$  in  $B_{k-1}$  because  $r \leq s_{k_r} \leq s_k$  and  $s_k$  is the distance between  $B_{k-1}^c$  and  $B_{k-\frac{5}{6}}$  (see (8)). Thus we have, for any  $1 \leq p \leq \infty$ ,

$$\begin{aligned} \|w'^2\|_{L^p(T_{k-1}, 0; L^p(B_{k-\frac{5}{6}}))} &= \left\| \left( u \frac{v_k}{|u|} \right) * \phi_r \right\|_{L^p(T_{k-1}, 0; L^p(B_{k-\frac{5}{6}}))} \\ &= \| |v_k| * \phi_r \|_{L^p(T_{k-1}, 0; L^p(B_{k-\frac{5}{6}}))} \leq \|v_k\|_{L^p(Q_{k-1})}. \end{aligned} \quad (58)$$

So, by using (43), we have

$$\|w'^2\|_{L^p(T_{k-1}, 0; L^p(B_{k-\frac{5}{6}}))} \leq C 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}}, \quad \text{for any } 1 \leq p \leq \frac{10}{3}. \quad (59)$$

*Remark 3.8.* The above computations says that, for any small  $r$ , the advection velocity  $w$  can be decomposed into one bounded part  $(w'^1 - w'')$  and the other part  $w'^2$ , which has a good contribution to the power of  $U_{k-1}$ .

Recall that the transpost term estimate (31) is valid for any  $0 < r < \infty$ . Moreover, the argument around (37) says that (31) holds even for the case  $r = 0$ . Thus, for any  $r \in [0, s_1)$ , we have the same pressure estimates (39), (40) and

(41). Thus we can follow the proof of the previous lemma 3.4 up to (48) without any single modification. It remains to control (I)–(IV).

For (I), (50) holds here too because (50) is independent of  $r$ .

For (II), by using (57) and (59) with the fact  $\text{supp}(\eta_k) \subset B_{k-\frac{1}{3}} \subset B_{k-\frac{5}{6}}$ , we have

$$\begin{aligned}
(II) &= \| |\nabla \eta_k| \cdot |w| \cdot |v_k|^2 \|_{L^1(Q_{k-1})} \\
&\leq C2^{3k} \left( \| (|w'^{1}| + |w''|) \cdot |v_k|^2 \|_{L^1(Q_{k-1})} + \| |w'^{2}| \cdot |v_k|^2 \|_{L^1(T_{k-1,0}; L^1(B_{k-\frac{5}{6}}))} \right) \\
&\leq C2^{3k} \|v_k\|_{L^2(Q_{k-1})}^2 + C2^{3k} \|w'^{2}\|_{L^{\frac{10}{3}}(T_{k-1,0}; L^{\frac{10}{3}}(B_{k-\frac{5}{6}}))} \cdot \| |v_k|^2 \|_{L^{\frac{10}{7}}(Q_{k-\frac{5}{6}})} \\
&\leq C2^{\frac{23k}{3}} U_{k-1}^{\frac{5}{3}} + C2^{10k} U_{k-1}^{\frac{5}{3}} \leq C2^{10k} U_{k-1}^{\frac{5}{3}}.
\end{aligned} \tag{60}$$

For (III)(non-local pressure term), we have (53) here too since (53) is independent of  $r$ .

For (IV)(local pressure term), by definition of  $P_{2,k}$  and decomposition of  $w$ ,

$$\begin{aligned}
-\Delta P_{2,k} &= \sum_{ij} \partial_i \partial_j \left( \psi_k w_i u_j \left(1 - \frac{v_k}{|u|}\right) + \psi_k w_i u_j \frac{v_k}{|u|} \right) \\
&= \sum_{ij} \partial_i \partial_j \left( \psi_k (w_i'^{1} - w_i'') u_j \left(1 - \frac{v_k}{|u|}\right) + \psi_k w_i'^{2} u_j \left(1 - \frac{v_k}{|u|}\right) \right. \\
&\quad \left. + \psi_k (w_i'^{1} - w_i'') u_j \frac{v_k}{|u|} + \psi_k w_i'^{2} u_j \frac{v_k}{|u|} \right).
\end{aligned}$$

Thus we can decompose  $P_{2,k}$  by

$$P_{2,k} = P_{2,k,1} + P_{2,k,2} + P_{2,k,3} + P_{2,k,4}$$

where

$$\begin{aligned}
P_{2,k,1} &= \sum_{ij} (\partial_i \partial_j) (-\Delta)^{-1} \left( \psi_k (w_i'^{1} - w_i'') u_j \left(1 - \frac{v_k}{|u|}\right) \right), \\
P_{2,k,2} &= \sum_{ij} (\partial_i \partial_j) (-\Delta)^{-1} \left( \psi_k w_i'^{2} u_j \left(1 - \frac{v_k}{|u|}\right) \right), \\
P_{2,k,3} &= \sum_{ij} (\partial_i \partial_j) (-\Delta)^{-1} \left( \psi_k (w_i'^{1} - w_i'') u_j \frac{v_k}{|u|} \right) \quad \text{and} \\
P_{2,k,4} &= \sum_{ij} (\partial_i \partial_j) (-\Delta)^{-1} \left( \psi_k w_i'^{2} u_j \frac{v_k}{|u|} \right).
\end{aligned}$$

By using  $\left| u \left(1 - \frac{v_k}{|u|}\right) \right| \leq 1$  and the fact  $\psi_k$  is supported in  $B_{k-\frac{5}{6}}$  with (57),

$$\|P_{2,k,1}\|_{L^p(T_{k-1,0}; L^p(\mathbb{R}^3))} \leq C_p, \quad \text{for } 1 < p < \infty \tag{61}$$

and, with (59),

$$\begin{aligned} \|P_{2,k,2}\|_{L^p(T_{k-1},0;L^p(\mathbb{R}^3))} &\leq C_p \| |\psi_k| \cdot |w'^{,2}| \|_{L^p(T_{k-1},0;L^p(\mathbb{R}^3))} \\ &\leq CC_p 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}} \quad \text{for } 1 \leq p \leq \frac{10}{3}. \end{aligned} \quad (62)$$

Observe that for  $i = 1, 2$ ,

$$\operatorname{div} \left( u G_i \right) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla G_i = \operatorname{div} \left( v_k \frac{u}{|u|} G_i \right) - G_i \operatorname{div} \left( \frac{u v_k}{|u|} \right). \quad (63)$$

For  $P_{2,k,1}$ , by using (42), (43), (45), (63) and (61) with  $p = 10$

$$\begin{aligned} &\int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u P_{2,k,1}) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P_{2,k,1} \right) (s, x) dx \right| ds \\ &\leq C^{3k} \|v_k \cdot |P_{2,k,1}|\|_{L^1(Q_{k-1})} + 3 \|d_k \cdot |P_{2,k,1}|\|_{L^1(Q_{k-1})} \\ &\leq C^{3k} \|v_k\|_{L^{\frac{10}{9}}(Q_{k-1})} \cdot \|P_{2,k,1}\|_{L^{10}(Q_{k-1})} + 3 \|d_k\|_{L^{\frac{10}{9}}(Q_{k-1})} \cdot \|P_{2,k,1}\|_{L^{10}(Q_{k-1})} \\ &\leq C 2^{\frac{16k}{3}} U_{k-1}^{\frac{3}{3}} + C 2^{\frac{5k}{3}} U_{k-1}^{\frac{7}{6}} \leq C 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}. \end{aligned} \quad (64)$$

Likewise, for  $P_{2,k,2}$ , by using (62) instead of (61)

$$\begin{aligned} &\int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u P_{2,k,2}) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla P_{2,k,2} \right) (s, x) dx \right| ds \\ &\leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{5}{3}} + C 2^{4k} U_{k-1}^{\frac{4}{3}} \leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{4}{3}}. \end{aligned} \quad (65)$$

From definitions of  $P_{2,k,3}$  and  $P_{2,k,4}$  with  $\operatorname{div}(w) = 0$ , we have

$$\begin{aligned} -\Delta \nabla (P_{2,k,3} + P_{2,k,4}) &= \sum_{ij} \partial_i \partial_j \nabla \left( \psi_k w_i u_j \frac{v_k}{|u|} \right) \\ &= \sum_{ij} \nabla \partial_j \left( (\partial_i \psi_k) w_i u_j \frac{v_k}{|u|} + \psi_k w_i \partial_i (u_j \frac{v_k}{|u|}) \right). \end{aligned}$$

Then we use the fact  $w = (w'^{,1} - w'') + w'^{,2}$  so that we can decompose

$$\nabla (P_{2,k,3} + P_{2,k,4}) = H_{1,k} + H_{2,k} + H_{3,k} + H_{4,k}$$

where

$$\begin{aligned} H_{1,k} &= \sum_{ij} (\nabla \partial_j) (-\Delta)^{-1} \left( (\partial_i \psi_k) (w_i'^{,1} - w_i'') u_j \frac{v_k}{|u|} \right), \\ H_{2,k} &= \sum_{ij} (\nabla \partial_j) (-\Delta)^{-1} \left( (\partial_i \psi_k) w_i'^{,2} u_j \frac{v_k}{|u|} \right), \\ H_{3,k} &= \sum_{ij} (\nabla \partial_j) (-\Delta)^{-1} \left( \psi_k (w_i'^{,1} - w_i'') \partial_i (u_j \frac{v_k}{|u|}) \right) \quad \text{and} \\ H_{4,k} &= \sum_{ij} (\nabla \partial_j) (-\Delta)^{-1} \left( \psi_k w_i'^{,2} \partial_i (u_j \frac{v_k}{|u|}) \right). \end{aligned}$$

By using  $|u| \leq 1 + v_k$ ,

$$\begin{aligned}
& \int_{T_{k-1}}^0 \left| \int_{\mathbb{R}^3} \eta_k(x) \left( \operatorname{div}(u(P_{2,k,3} + P_{2,k,4})) + \left( \frac{v_k}{|u|} - 1 \right) u \cdot \nabla(P_{2,k,3} + P_{2,k,4}) \right) dx \right| ds \\
& \leq C^{3k} \int_{Q_{k-1}} (1 + v_k) \cdot |(P_{2,k,3} + P_{2,k,4})(s, x)| + |\nabla(P_{2,k,3} + P_{2,k,4})| dx ds \\
& \leq C^{3k} \left( \|P_{2,k,3}\|_{L^1(Q_{k-1})} + \|v_k \cdot |P_{2,k,3}|\|_{L^1(Q_{k-1})} \right. \\
& \quad + \|P_{2,k,4}\|_{L^1(Q_{k-1})} + \|v_k \cdot |P_{2,k,4}|\|_{L^1(Q_{k-1})} \\
& \quad \left. + \|H_{1,k}\|_{L^1(Q_{k-1})} + \|H_{2,k}\|_{L^1(Q_{k-1})} + \|H_{3,k}\|_{L^1(Q_{k-1})} + \|H_{4,k}\|_{L^1(Q_{k-1})} \right). \tag{66}
\end{aligned}$$

From (43) and (57) with the Riesz transform,

$$\|P_{2,k,3}\|_{L^1(Q_{k-1})} \leq C \|P_{2,k,3}\|_{L^{\frac{10}{9}}(T_{k-1}, 0; L^{\frac{10}{9}}(\mathbb{R}^3))} \leq C \|v_k\|_{L^{\frac{10}{9}}(Q_{k-1})} \leq C 2^{\frac{7k}{3}} U_{k-1}^{\frac{3}{2}}. \tag{67}$$

Likewise

$$\|H_{1,k}\|_{L^1(Q_{k-1})} \leq C 2^{\frac{16k}{3}} U_{k-1}^{\frac{3}{2}} \tag{68}$$

and

$$\begin{aligned}
\|v_k \cdot |P_{2,k,3}|\|_{L^1(Q_{k-1})} & \leq \|v_k\|_{L^2(Q_{k-1})} \|P_{2,k,3}\|_{L^2(Q_{k-1})} \\
& \leq C 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{6}} \cdot C 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{6}} \leq C 2^{\frac{14k}{3}} U_{k-1}^{\frac{5}{3}}. \tag{69}
\end{aligned}$$

Using (43), (59), (42) and (45), we have

$$\|P_{2,k,4}\|_{L^1(Q_{k-1})} \leq C 2^{\frac{14k}{3}} U_{k-1}^{\frac{3}{2}}, \tag{70}$$

$$\|H_{2,k}\|_{L^1(Q_{k-1})} \leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{3}{2}}, \tag{71}$$

$$\|v_k \cdot |P_{2,k,4}|\|_{L^1(Q_{k-1})} \leq C 2^{\frac{21k}{3}} U_{k-1}^{\frac{5}{3}}, \tag{72}$$

$$\|H_{3,k}\|_{L^1(Q_{k-1})} \leq C 2^{\frac{5k}{3}} U_{k-1}^{\frac{7}{6}} \tag{73}$$

and

$$\|H_{4,k}\|_{L^1(Q_{k-1})} \leq C 2^{4k} U_{k-1}^{\frac{7}{6}}. \tag{74}$$

Combining (64), (65) and (66) together with (67),  $\dots$ , (74), we obtain

$$(IV) \leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{7}{6}}. \tag{75}$$

Finally we combine (60) and (75) together with (50) and (53) in the previous lemma in order to finish the proof of this lemma 3.5.  $\square$

### 3.4 Combining the two De Giorgi arguments

First we present one small lemma. Then the actual proof of the proposition 2.1 will follow. The following small lemma says that certain non-linear estimates give zero limit if the initial term is sufficiently small. This fact is one of key arguments of De Giorgi method.

**Lemma 3.6.** *Let  $C > 1$  and  $\beta > 1$ . Then there exists a constant  $C_0^*$  such that for every sequence verifying both  $0 \leq W_0 < C_0^*$  and*

$$0 \leq W_k \leq C^k \cdot W_{k-1}^\beta \quad \text{for any } k \geq 1,$$

*we have  $\lim_{k \rightarrow \infty} W_k = 0$ .*

*Proof.* It is quite standard or see the lemma 1 in [42]. □

Finally we are ready to prove the proposition 2.1.

*Proof of proposition 2.1.* Suppose that  $u$  is a solution of (Problem II-r) for some  $0 \leq r < \infty$  verifying

$$\begin{aligned} & \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \leq \delta \\ & \text{and } \|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^2(B(2)))} \leq \delta \end{aligned}$$

where  $\delta$  will be chosen within the proof.

From two big lemmas 3.4 and 3.5 by assuming  $\delta \leq \min(\delta_1, \delta_2)$ , we have

$$U_k \leq \begin{cases} (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r \geq s_1. \\ \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } 0 < r < s_1. \\ (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} & \text{for } k = 1, 2, \dots, k_r \quad \text{if } 0 \leq r < s_1. \end{cases} \quad (76)$$

Note that  $k_r = \infty$  if  $r = 0$ . Thus we can combine the case  $r = 0$  with the case  $r \geq s_1$  into one estimate:

$$U_k \leq (\bar{C}_3)^k U_{k-1}^{\frac{7}{6}} \quad \text{for any } k \geq 1 \quad \text{if either } r \geq s_1 \text{ or } r = 0.$$

where we define  $\bar{C}_3 = \max(\bar{C}_1, \bar{C}_2)$ .

We consider now the case  $0 < r < s_1$ . Recall that  $s_k = D \cdot 2^{-3k}$  where  $D = ((\sqrt{2} - 1)2\sqrt{2}) > 1$  and  $s_{k_r+1} < r \leq s_{k_r}$  for any  $r \in (0, s_1)$ . It gives us  $r \geq D \cdot 2^{-3(k_r+1)}$ . Thus if  $k \geq k_r$  and if  $0 < r < s_1$ , then the second line in (76) becomes

$$\begin{aligned} U_k & \leq \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \leq \frac{2^{9(k_r+1)}}{D^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \\ & \leq 2^{9(k+1)} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \leq (2^{18} \cdot \bar{C}_1)^k U_{k-1}^{\frac{7}{6}}. \end{aligned} \quad (77)$$

So we have for any  $r \in (0, s_1)$ ,

$$U_k \leq \begin{cases} (2^{18} \cdot \bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq k_r. \\ (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} & \text{for } k = 1, 2, \dots, k_r. \end{cases}$$

Define  $\bar{C} = \max(2^{18} \cdot \bar{C}_1, \bar{C}_2, \bar{C}_3) = \max(2^{18} \cdot \bar{C}_1, \bar{C}_2)$ . Then we can combine all three cases  $r = 0$ ,  $0 < r < s_1$  and  $s_1 \leq r < \infty$  into one uniform estimate:

$$U_k \leq (\bar{C})^k U_{k-1}^{\frac{7}{6}} \quad \text{for any } k \geq 1 \quad \text{and for any } 0 \leq r < \infty.$$

Finally, by using the recursive lemma 3.6, we obtain  $C_0^*$  such that  $U_k \rightarrow 0$  as  $k \rightarrow 0$  whenever  $U_0 < C_0^*$ . This condition  $U_0 < C_0^*$  is achievable once we assume  $\delta$  so small that  $\delta \leq \sqrt{\frac{C_0^*}{2}}$  because

$$U_0 \leq \left( \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \right)^2.$$

Thus we fix  $\delta = \min(\sqrt{\frac{C_0^*}{2}}, \delta_1, \delta_2)$  which does not depend on any  $r \in [0, \infty)$ . Observe that for any  $k \geq 1$ ,

$$\sup_{-\frac{3}{2} \leq t \leq 0} \int_{B(\frac{1}{2})} (|u(t, x)| - 1)_+^2 dx \leq U_k$$

from  $E_k \leq 1$  and  $(-\frac{3}{2}, 0) \times B(\frac{1}{2}) \subset Q_k$ . Due to the fact  $U_k \rightarrow 0$ , the conclusion of this proposition 2.1 follows.  $\square$

## 4 Proof of the second local study proposition 2.2

First we present technical lemmas, whose proofs will be given in the appendix. In the subsection 4.2, it will be explained how to apply the previous local study proposition 2.1 in order to get a  $L^\infty$ -bound of the velocity  $u$ . Then, the subsections 4.3 and 4.4 will give us  $L^\infty$ -bounds for classical derivatives  $\nabla^d u$  and for fractional derivatives  $(-\Delta)^{\alpha/2} \nabla^d u$ , respectively.

### 4.1 Some lemmas

The following lemma is an estimate about higher derivatives of pressure which we can find a similar lemma in [41]. However they are different in the sense that here we require  $(n-1)$ th order norm of  $v_1$  to control  $n$ th derivatives of pressure (see (78)) while in [41] we require one more order, i.e.  $n$ th order. This fact follows the divergence structure and it will be useful for a bootstrap argument in the subsection 4.3 when large  $r$  is large (we will see (84)).

**Lemma 4.1.** *Suppose that we have  $v_1, v_2 \in (C^\infty(B(1)))^3$  with  $\operatorname{div} v_1 = \operatorname{div} v_2 = 0$  and  $P \in C^\infty(B(1))$  which satisfy*

$$-\Delta P = \operatorname{div} \operatorname{div}(v_2 \otimes v_1)$$

on  $B(1) \subset \mathbb{R}^3$ .

Then, for any  $n \geq 2$ ,  $0 < b < a < 1$  and  $1 < p < \infty$ , we have the two following estimates:

$$\begin{aligned} \|\nabla^n P\|_{L^p(B(b))} &\leq C_{(a,b,n,p)} \left( \|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))} \right. \\ &\quad \left. + \|P\|_{L^1(B(a))} \right) \end{aligned} \quad (78)$$

where  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ , and

$$\begin{aligned} \|\nabla^n P\|_{L^\infty(B(b))} &\leq C_{(a,b,n)} \left( \|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))} \right. \\ &\quad \left. + \|P\|_{L^1(B(a))} \right) \end{aligned} \quad (79)$$

Note that such constants are independent of any  $v_1, v_2$  and  $P$ . Also,  $\infty$  is allowed for  $p_1$  and  $p_2$ . e.g. if  $p_1 = \infty$ , then  $p_2 = p$ .

*Proof.* See the appendix.  $\square$

The following is a local result by using a parabolic regularization. It will be used in the subsection 4.3 to prove (82) and (84).

**Lemma 4.2.** *Suppose that we have smooth solution  $(v_1, v_2, P)$  on  $Q(1) = (-1, 0) \times B(1)$  of*

$$\begin{aligned} \partial_t(v_1) + \operatorname{div}(v_2 \otimes v_1) + \nabla P - \Delta v_1 &= 0 \\ \operatorname{div}(v_1) &= 0 \text{ and } \operatorname{div}(v_2) = 0. \end{aligned}$$

Then, for any  $n \geq 1$ ,  $0 < b < a < 1$ ,  $1 < p_1 < \infty$  and  $1 < p_2 < \infty$ , we have

$$\begin{aligned} \|\nabla^n v_1\|_{L^{p_1}(-(b)^2, 0; L^{p_2}(B(b)))} &\leq C_{(a,b,n,p_1,p_2)} \left( \|v_2 \otimes v_1\|_{L^{p_1}(-a^2, 0; W^{n-1,p_2}(B(a)))} \right. \\ &\quad \left. + \|v_1\|_{L^{p_1}(-a^2, 0; W^{n-1,p_2}(B(a)))} + \|P\|_{L^1(-a^2, 0; L^1(B(a)))} \right) \end{aligned} \quad (80)$$

where  $v_2 \otimes v_1$  is the matrix whose  $(i, j)$  component is the product of  $j$ -th component  $v_{2,j}$  of  $v_2$  and  $i$ -th one  $v_{1,i}$  of  $v_1$  and  $\left( \operatorname{div}(v_2 \otimes v_1) \right)_i = \sum_j \partial_j(v_{2,j} v_{1,i})$ .

Note that such constants are independent of any  $v_1, v_2$  and  $P$ .



Proof of this lemma 4.2 is omitted because it is based on the standard parabolic regularization result (e.g. Solonnikov [38]) and precise argument is essentially contained in [41] except that here we consider

$$(v_1)_t + \operatorname{div}(v_2 \otimes v_1) + \nabla P - \Delta v_1 = 0$$

while [41] covered

$$(u)_t + \operatorname{div}(u \otimes u) + \nabla P - \Delta u = 0.$$

The following lemma will be used in the subsection 4.3, especially when we prove (84) for large  $r$ .

**Lemma 4.3.** *Suppose that we have smooth solution  $(v_1, v_2, P)$  on  $Q(1) = (-1, 0) \times B(1)$  of*

$$\begin{aligned} \partial_t(v_1) + (v_2 \cdot \nabla)(v_1) + \nabla P - \Delta v_1 &= 0 \\ \operatorname{div}(v_1) &= 0 \text{ and } \operatorname{div}(v_2) = 0. \end{aligned}$$

*Then, for any  $n \geq 0$  and  $0 < b < a < 1$ , we have*

$$\begin{aligned} \|\nabla^n v_1\|_{L^\infty(-(b)^2, 0; L^1(B(b)))} &\leq \\ C_{(a,b,n)} \Big[ &\left( \|v_2\|_{L^2(-a^2, 0; W^{n,\infty}(B(a)))} + 1 \right) \cdot \|v_1\|_{L^2(-a^2, 0; W^{n,1}(B(a)))} \\ &+ \|\nabla^{n+1} P\|_{L^1(-a^2, 0; L^1(B(a)))} \Big] \end{aligned}$$

*and, for any  $p \geq 1$ ,*

$$\begin{aligned} \|\nabla^n v_1\|_{L^\infty(-(b)^2, 0; L^{p+\frac{1}{2}}(B(b)))}^{p+\frac{1}{2}} &\leq \\ C_{(a,b,n,p)} \Big[ &\left( \|v_2\|_{L^2(-a^2, 0; W^{n,\infty}(B(a)))} + 1 \right) \cdot \|v_1\|_{L^2(-(a)^2, 0; W^{n,2p}(B(a)))} \\ &+ \|\nabla^{n+1} P\|_{L^1(-(a)^2, 0; L^{2p}(B(a)))} \Big] \cdot \|v_1\|_{L^\infty(-(a)^2, 0; W^{n,p}(B(a)))}^{p-\frac{1}{2}}. \end{aligned}$$

*Note that such constants are independent of any  $v_1, v_2$  and  $P$ .*

*Proof.* See the appendix. □

The following non-local version of Sobolev-type lemmas will be useful when we handle fractional derivatives by Maximal functions. We will see in the subsection 4.4 that the power  $(1 + \frac{3}{p})$  of  $M$  on the right hand side of the following estimate is very important to obtain a required estimate (90).

**Lemma 4.4.** *Let  $M_0 > 0$  and  $1 \leq p < \infty$ . Then there exist  $C = C(M_0, p)$  with the following property:*

*For any  $M \geq M_0$  and for any  $f \in C^1(\mathbb{R}^3)$  such that  $\int_{\mathbb{R}^3} \phi(x) f(x) dx = 0$ , we have*

$$\|f\|_{L^p(B(M))} \leq CM^{1+\frac{3}{p}} \cdot \left( \|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p} + \|\nabla f\|_{L^1(B(2))} \right).$$

*Proof.* See the appendix.  $\square$

With the above lemmas, we are ready to prove the proposition 2.2.

*Proof of proposition 2.2.* We divide this proof into three stages.

Stage 1 in subsection 4.2: First, we will obtain a  $L_t^\infty L_x^2$ -local bound for  $u$  by using the mean-zero property of  $u$  and  $w$ . Then, a  $L^\infty$ -local bound of  $u$  follows thanks to the first local study proposition 2.1.

Stage 2 in subsection 4.3: We will get a  $L^\infty$ -local bound for  $\nabla^d u$  for  $d \geq 1$  by using an induction argument with a boot-strapping. This is not obvious especially when  $r$  is large because  $w$  depends a non-local part of  $u$  while our knowledge about the  $L^\infty$ -bound of  $u$  from the stage 1 is only local.

Stage 3 in subsection 4.4: We will achieve a  $L^\infty$ -local bound for  $(-\Delta)^{\alpha/2} \nabla^d u$  for  $d \geq 1$  with  $0 < \alpha < 2$  from the integral representation of the fractional Laplacian. The non-locality of this fractional operator will let us to adopt more complicated conditions (see (23)).

## 4.2 Stage 1: to obtain $L^\infty$ -local bound for $u$ .

First we suppose that  $u$  satisfies all conditions of the proposition 2.2 without (23) (The condition (23) will be assumed only at the stage 3). Our goal is to find a sufficiently small  $\bar{\eta}$  which is independent of  $r \in [0, \infty)$ .

Assume  $\bar{\eta} \leq 1$  and define  $\bar{r}_0 = \frac{1}{4}$  for this subsection. From (21), we get

$$\|u\|_{L^2(-4,0;L^6(B(2)))} \leq C \|\nabla u\|_{L^2(-4,0;L^2(B(2)))} \leq C \cdot \bar{\eta}.$$

From the corollary 3.2, if  $r \geq \bar{r}_0$ , then

$$\|w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \cdot \bar{\eta}.$$

On the other hand, if  $0 \leq r < \bar{r}_0$ , then

$$\|w'\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \leq C \|u\|_{L^2(-4,0;L^6(B(2)))} \leq C \bar{\eta}$$

because  $\phi_r$  is supported in  $B(r) \subset B(\bar{r})$ , and  $w = u * \phi_r$  (see (8)). For  $w''$ ,

$$\begin{aligned} \|w''\|_{L^2(-4,0;L^\infty(B(2)))} &\leq \|u * \phi_r\|_{L^1(B(1))} \|L^2((-4,0))\| \leq \|u\|_{L^1(B(2))} \|L^2((-4,0))\| \\ &\leq C \|u\|_{L^2(-4,0;L^6(B(2)))} \leq C \bar{\eta}. \end{aligned}$$

Thus  $\|w\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \leq C \bar{\eta}$  if  $r < \bar{r}_0$  from  $w = w' + w''$ .

In sum, for any  $0 \leq r < \infty$ ,

$$\|w\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \leq C \bar{\eta}. \quad (81)$$

Since the equation (18) depends only on  $\nabla P$ , without loss of generality, we may assume  $\int_{\mathbb{R}^3} \phi(x) P(t, x) = 0$  for  $t \in (-4, 0)$ . Then with the mean zero property (21) of  $u$ , we have

$$\left\| \int_{\mathbb{R}^3} \phi(x) \nabla P(\cdot, x) dx \right\|_{L^1(-4, 0)} \leq C \bar{\eta}^{\frac{1}{2}}$$

after integration in  $x$ .

From Sobolev,

$$\begin{aligned} \|\nabla P\|_{L^1(-4, 0; L^{\frac{3}{2}}(B(\frac{7}{4})))} &\leq C \bar{\eta}^{\frac{1}{2}} \\ \text{and } \|P\|_{L^1(-4, 0; L^3(B(\frac{7}{4})))} &\leq C \bar{\eta}^{\frac{1}{2}} \end{aligned}$$

Then we follow step 1 and step 2 of the proof of the proposition 10 in [41], we can obtain

$$\|u\|_{L^\infty(-3, 0; L^{\frac{3}{2}}(B(\frac{6}{4})))} \leq C \bar{\eta}^{\frac{1}{3}}.$$

and then

$$\|u\|_{L^\infty(-2, 0; L^2(B(\frac{5}{4})))} \leq C \bar{\eta}^{\frac{1}{4}}$$

for  $0 \leq r < \infty$ . Details are omitted.

Finally, by taking  $0 < \bar{\eta} < 1$  such that  $C \bar{\eta}^{\frac{1}{4}} \leq \bar{\delta}$ , we have all assumptions of the proposition 2.1. As a result, we have  $|u(t, x)| \leq 1$  on  $[-\frac{3}{2}, 0] \times B(\frac{1}{2})$ .

### 4.3 Stage 2: to obtain $L^\infty$ local bound for $\nabla^d u$ .

Here we cover only classical derivatives, i.e.  $\alpha = 0$ . For any integer  $d \geq 1$ , our goal is to find  $C_{d,0}$  such that  $|((-\Delta)^{\frac{d}{2}} \nabla^d) u(t, x)| = |\nabla^d u(t, x)| \leq C_{d,0}$  on  $(-\frac{1}{3})^2, 0) \times (B(\frac{1}{3}))$ .

We define a strictly decreasing sequence of balls and parabolic cylinders from  $(-\frac{1}{2})^2, 0) \times B(\frac{1}{2})$  to  $(-\frac{1}{3})^2, 0) \times (B(\frac{1}{3}))$  by

$$\begin{aligned} \bar{B}_n &= B(\frac{1}{3} + \frac{1}{6} \cdot 2^{-n}) = B(l_n) \\ \bar{Q}_n &= (-\frac{1}{3} + \frac{1}{6} \cdot 2^{-n})^2, 0) \times \bar{B}_n = (-l_n)^2, 0) \times \bar{B}_n \end{aligned}$$

where  $l_n = \frac{1}{3} + \frac{1}{6} \cdot 2^{-n}$ .

First we claim in order to cover the small  $r$  case:

There exist two positive sequences  $\{\bar{r}_n\}_{n=0}^\infty$  and  $\{C_{n,small}\}_{n=0}^\infty$  such that for any integer  $n \geq 0$  and for any  $r \in [0, \bar{r}_n)$ ,

$$\|\nabla^n u\|_{L^\infty(\bar{Q}_{11n})} \leq C_{n,small}. \quad (82)$$

Indeed, from the previous subsection 4.2 (the stage 1), (82) holds for  $n = 0$  by taking  $\bar{r}_0 = 1$  and  $C_{0,small} = 1$ . We define  $\bar{r}_n$  = distance between  $B_{11n}$  and  $(B_{11n-1})^c$  for  $n \geq 1$ . Then  $\{\bar{r}_n\}_{n=0}^\infty$  is decreasing to zero as  $n$  goes to  $\infty$ . Moreover, we can control  $w$  by  $u$  as long as  $0 \leq r < \bar{r}_n$ : for any  $n \geq 1$ ,

$$\begin{aligned} \|w\|_{L^{p_1}(-(l_m)^2, 0; L^{p_2}(\bar{B}_m))} &\leq \left( \|u\|_{L^{p_1}(-(l_{m-1})^2, 0; L^{p_2}(\bar{B}_{m-1}))} + C \right) \quad \text{and} \\ \|\nabla^k w\|_{L^{p_1}(-(l_m)^2, 0; L^{p_2}(\bar{B}_m))} &\leq \|\nabla^k u\|_{L^{p_1}(-(l_{m-1})^2, 0; L^{p_2}(\bar{B}_{m-1}))} \end{aligned} \quad (83)$$

for any integer  $m$  such that  $m \leq 11 \cdot n$ , for any  $k \geq 1$  and for any  $p_1 \in [1, \infty]$  and  $p_2 \in [1, \infty]$  (see (8)).

We will use an induction with a boot-strapping. First we fix  $d \geq 1$  and suppose that (82) is true up to  $n = (d-1)$ . It implies for any  $r \in [0, \bar{r}_{d-1})$

$$\|u\|_{L^\infty(-l_s^2, 0; W^{d-1, \infty}(\bar{B}_s))} \leq C$$

where  $s = 11(d-1)$ . We want to show that (82) is also true for the case  $n = d$ .

From (83),  $\|w\|_{L^\infty(-l_{s+1}^2, 0; W^{d-1, \infty}(\bar{B}_{s+1}))} \leq C$  and, From the lemma 4.2 with  $v_2 = w$  and  $v_1 = u$ ,  $\|u\|_{L^{16}(-l_{s+2}^2, 0; W^{d, 32}(\bar{B}_{s+2}))} \leq C$ . Then, we use (83) and the lemma 4.2 in turn:

$$\begin{aligned} &\rightarrow w \in L^{16}(-l_{s+3}^2, 0; W^{d, 32}(\bar{B}_{s+3})) \rightarrow u \in L^8(-l_{s+4}^2, 0; W^{d+1, 16}(\bar{B}_{s+4})) \\ &\rightarrow w \in L^8 W^{d+1, 16} \rightarrow \dots \rightarrow u \in L^2 W^{d+3, 4} \end{aligned}$$

Then, from Sobolev,

$$\rightarrow u \in L^2 W^{d+2, \infty} \rightarrow w \in L^2(-l_{s+9}^2, 0; W^{d+2, \infty}(\bar{B}_{s+9})).$$

This estimate gives us

$$\Delta(\nabla^d u), \operatorname{div}(\nabla^d(w \otimes u)) \text{ and } \nabla(\nabla^d P) \in L^1(-l_{s+10}^2, 0; L^\infty(\bar{B}_{s+10}))$$

where we used (79) for the pressure term. Thus

$$\partial_t(\nabla^d u) \in L^1(-l_{s+10}^2, 0; L^\infty(\bar{B}_{s+10})).$$

Finally, we obtain that for any  $r \in [0, \bar{r}_d)$

$$\|\nabla^d u\|_{L^\infty(-l_{s+11}^2, 0; L^\infty(\bar{B}_{s+11}))} \leq C.$$

where  $C$  depends only on  $d$ . By the induction argument, we showed the above claim (82).

Now we introduce the second claim:

There exist a sequences  $\{C_{n,large}\}_{n=0}^\infty$  such that for any integer  $n \geq 0$  and for any  $r \geq \bar{r}_n$ ,

$$\|\nabla^n u\|_{L^\infty(\bar{Q}_{21 \cdot n})} \leq C_{n,large} \quad (84)$$

where  $\bar{r}_n$  comes from previous claim (82).

Before proving the above second claim (84), we need the following two observations **(I)**, **(II)** from the lemmas 4.2 and 4.1:

**(I).** From the corollary 3.2 for any  $n \geq 0$ , if  $r \geq \bar{r}_n$ , then

$$\|w\|_{L^2(-4,0;W^{n,\infty}(B(2)))} \leq C_n.$$

We use (80) in the lemma 4.2 with  $v_1 = u$  and  $v_2 = w$ . Then it becomes

$$\|\nabla^n u\|_{L^{p_1}(-(l_m)^2,0;L^{p_2}(\bar{B}_m))} \leq C_{(m,n,p_2)} \left( \|u\|_{L^{\frac{2p_1}{2-p_1}}(-(l_{m-1})^2,0;W^{n-1,p_2}(\bar{B}_{m-1}))} + 1 \right) \quad (85)$$

for  $n \geq 1$ ,  $m \geq 1$ ,  $1 < p_1 \leq 2$  and  $1 < p_2 < \infty$ . (For the case  $p_1 = 2$ , we may interpret  $\frac{2p_1}{2-p_1} = \infty$ .)

**(II).** Moreover, (78) in the lemma 4.1 becomes

$$\|\nabla^n P\|_{L^1(-(l_m)^2,0;L^p(\bar{B}_m))} \leq C_{(m,n,p)} \left( \|u\|_{L^2(-(l_{m-1})^2,0;W^{n-1,p}(\bar{B}_{m-1}))} + 1 \right) \quad (86)$$

for  $n \geq 2$  and  $1 < p < \infty$ .

Now we are ready to prove the second claim (84) by an induction with a boot-strapping. From the previous subsection 4.2 (the stage 1), (84) holds for  $n = 0$  with  $C_{0,large} = 1$ . Fix  $d \geq 1$  and suppose that we have (84) up to  $n = (d-1)$ . It implies for any  $r \geq \bar{r}_{d-1}$

$$\|u\|_{L^\infty(-l_s^2,0;W^{d-1,\infty}(\bar{B}_s))} \leq C_{d-1,large}$$

where  $s = 21(d-1)$ . We want to show (84) for  $n = d$ .

By using (85) with  $n = d$ ,  $p_1 = 2$  and  $p_2 = 11$ ,

$$\|u\|_{L^2(-l_{s+1}^2,0;W^{d,11}(\bar{B}_{s+1}))} \leq C$$

and, from (86) with  $n = d+1$ ,  $m = 0$  and  $p = 11$ ,

$$\|\nabla^{d+1} P\|_{L^1(-l_{s+2}^2,0;L^{11}(\bar{B}_{s+2}))} \leq C.$$

Combining the above two results with the lemma 4.3 for  $v_1 = u$  and  $v_2 = w$ , we can have increased integrability in space by 0.5 up to 6:

$$\begin{aligned} \|u\|_{L^\infty(-l_{s+3}^2, 0; W^{d,1}(\bar{B}_{s+3}))} &\leq C, \\ \|u\|_{L^\infty(-l_{s+4}^2, 0; W^{d,1.5}(\bar{B}_{s+4}))} &\leq C, \\ &\dots, \quad \text{and} \\ \|u\|_{L^\infty(-l_{s+13}^2, 0; W^{d,6}(\bar{B}_{s+13}))} &\leq C. \end{aligned}$$

By using (85) and (86) again, we have

$$\begin{aligned} \|u\|_{L^2(-l_{s+14}^2, 0; W^{d+1,6}(\bar{B}_{s+14}))} &\leq C \quad \text{and} \\ \|\nabla^{d+2} P\|_{L^1(-l_{s+15}^2, 0; L^6(\bar{B}_{s+15}))} &\leq C. \end{aligned}$$

Combining the above two results with the lemma 4.3 again, we have

$$\begin{aligned} \|u\|_{L^\infty(-l_{s+16}^2, 0; W^{d+1,1}(\bar{B}_{s+16}))} &\leq C, \\ &\dots, \quad \text{and} \\ \|u\|_{L^\infty(-l_{s+21}^2, 0; W^{d+1,3.5}(\bar{B}_{s+21}))} &\leq C. \end{aligned}$$

Finally, from Sobolev's inequality,

$$\|\nabla^d u\|_{L^\infty(-l_{s+21}^2, 0; L^\infty(\bar{B}_{s+21}))} \leq C$$

where  $C$  depends only  $d$  not  $u$  nor  $r$  as long as  $r \geq \bar{r}_d$ . From induction, we proved second claim (84).

Define for any  $n \geq 0$ ,  $C_{n,0} = \max(C_{n,small}, C_{n,large})$  where  $C_{n,small}$  and  $C_{n,large}$  come from (82) and (84) respectively. Then we have:

$$\|\nabla^n u\|_{L^\infty(Q(\frac{1}{3}))} \leq C_{n,0} \tag{87}$$

for any  $n \geq 0$  and for any  $0 \leq r < \infty$  because  $Q(\frac{1}{3}) \subset \bar{Q}_n$ . It ends this stage 2.

#### 4.4 Stage 3: to obtain $L^\infty$ local bound for $(-\Delta)^{\alpha/2} \nabla^d u$ .

From now on, we assume further that  $(u, P)$  satisfies (23) as well as all the other conditions of the proposition 2.2. In the following proof, we will not divide the proof into a small  $r$  part and a large  $r$  part.

Fix an integer  $d \geq 1$  and a real  $\alpha$  with  $0 < \alpha < 2$ . i.e. any constant which will appear may depend  $d$  and  $\alpha$ . But they will be independent of any  $r \in [0, \infty)$  and any solution  $(u, P)$ .

First, we claim:

There exists a constant  $C = C(d, \alpha)$  such that

$$|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u(t, x)| \leq C(d, \alpha) + \left| \int_{|y| \geq (1/6)} \frac{\nabla^d u(t, x-y)}{|y|^{3+\alpha}} dy \right| \quad (88)$$

for  $|x| \leq (1/6)$  and for  $-(1/3)^2 \leq t \leq 0$ .

To prove (88), we first recall the Taylor expansion of any  $C^2$  function  $f$  at  $x$ :  $f(y) - f(x) = (\nabla f)(x) \cdot (y - x) + R(x, y)$ , and we have an error estimate  $|R| \leq C|x - y|^2 \cdot \|\nabla^2 f\|_{L^\infty(B(x; |x-y|))}$ . Note that if we integrate the first order term  $(\nabla f)(x) \cdot (y - x)$  in  $y$  on any sphere with the center  $x$ , we have zero by symmetry. As a result, if we take any  $x$  and  $t$  for  $|x| \leq (1/6)$  and for  $-(1/3)^2 \leq t \leq 0$  respectively, then we have

$$\begin{aligned} |(-\Delta)^{\frac{\alpha}{2}} \nabla^d u(t, x)| &= \left| P.V. \int_{\mathbb{R}^3} \frac{\nabla^d u(t, x) - \nabla^d u(t, y)}{|x - y|^{3+\alpha}} dy \right| \\ &\leq \sup_{z \in B((1/3))} (|\nabla^{d+2} u(t, z)|) \cdot \int_{|x-y| < (1/6)} \frac{1}{|x - y|^{3+\alpha-2}} dy \\ &\quad + \sup_{z \in B((1/3))} (|\nabla^d u(t, z)|) \cdot \int_{|x-y| \geq (1/6)} \frac{1}{|x - y|^{3+\alpha}} dy \\ &\quad + \left| \int_{|x-y| \geq (1/6)} \frac{\nabla^d u(t, y)}{|x - y|^{3+\alpha}} dy \right| \\ &\leq C(d, \alpha) + \left| \int_{|y| \geq (1/6)} \frac{\nabla^d u(t, x-y)}{|y|^{3+\alpha}} dy \right| \end{aligned}$$

where we used the result (87) of the previous subsection 4.3 (the stage 2) together with the Taylor expansion of  $\nabla^d u(t, \cdot)$  at  $x$  in order to reduce singularity by 2 at the origin  $x = y$ . We proved the first claim (88).

Second, we claim:

There exists  $C = C(d, \alpha)$  such that

$$\left| \int_{|y| \geq (1/6)} \frac{\nabla^d u(t, x-y)}{|y|^{3+\alpha}} dy \right| \leq C(d, \alpha) + \sum_{j=k}^{\infty} \left( \frac{1}{2^\alpha} \right)^j \cdot |((h^\alpha)_{2^j} * \nabla^d u)(t, x)| \quad (89)$$

for  $|x| \leq (1/6)$  and for  $-(1/3)^2 \leq t \leq 0$  where  $k$  is the integer such that  $2^k \leq (1/6) < 2^{k+1}$ . (i.e. from now on, we fix  $k = -3$ ). Recall that  $h^\alpha$  is defined around (12).

To prove the above second claim (89): (Recall (11) and (12))

$$\begin{aligned}
 & \left| \int_{|y| \geq (1/6)} \frac{\nabla^d u(t, x-y)}{|y|^{3+\alpha}} dy \right| = \left| \int_{|y| \geq (1/6)} \sum_{j=k}^{\infty} \zeta\left(\frac{y}{2^j}\right) \frac{\nabla^d u(t, x-y)}{|y|^{3+\alpha}} dy \right| \\
 &= \left| \int_{|y| \geq (1/6)} \sum_{j=k}^{\infty} \frac{1}{(2^j)^\alpha} \cdot (h^\alpha)_{2^j}(y) \nabla^d u(t, x-y) dy \right| \\
 &\leq \sum_{j=k}^{k+1} \frac{1}{(2^j)^\alpha} \cdot \left| \int_{|y| \geq (1/6)} (h^\alpha)_{2^j}(y) \nabla^d u(t, x-y) dy \right| \\
 &\quad + \sum_{j=k+2}^{\infty} \frac{1}{(2^j)^\alpha} \cdot \left| \int_{|y| \geq (1/6)} (h^\alpha)_{2^j}(y) \nabla^d u(t, x-y) dy \right| \\
 &= (I) + (II).
 \end{aligned}$$

For (I),

$$\begin{aligned}
 (I) &\leq \sum_{j=k}^{k+1} \frac{1}{(2^j)^\alpha} \cdot \left( \left| \int_{\mathbb{R}^3} (h^\alpha)_{2^j}(y) \nabla^d u(t, x-y) dy \right| \right. \\
 &\quad \left. + \int_{|y| \leq (1/6)} |(h^\alpha)_{2^j}(y)| \cdot |\nabla^d u(t, x-y)| dy \right) \\
 &\leq \sum_{j=k}^{k+1} \frac{1}{(2^j)^\alpha} \left( |((h^\alpha)_{2^j} * \nabla^d u)(t, x)| + C \cdot \sup_{z \in B(1/3)} |\nabla^d u(t, z)| \right) \\
 &= \sum_{j=k}^{k+1} \left( \frac{1}{2^\alpha} \right)^j \cdot |((h^\alpha)_{2^j} * \nabla^d u)(t, x)| + C(d, \alpha).
 \end{aligned}$$

For (II), by using  $\text{supp}(h_{2^j}^\alpha) \subset (B(2^{j-1}))^C \subset (B(1/6))^C$  for any  $j \geq k+2$ ,

$$\begin{aligned}
 (II) &= \sum_{j=k+2}^{\infty} \frac{1}{(2^j)^\alpha} \cdot \left| \int_{\mathbb{R}^3} (h^\alpha)_{2^j}(y) \nabla^d u(t, x-y) dy \right| \\
 &= \sum_{j=k+2}^{\infty} \left( \frac{1}{2^\alpha} \right)^j \cdot |((h^\alpha)_{2^j} * \nabla^d u)(t, x)|.
 \end{aligned}$$

We showed the second claim (89).

Third, we claim:

There exists  $C = C(d, \alpha)$  such that

$$\| (h^\alpha)_M * \nabla^d u \|_{L^\infty(-(1/6)^2, 0; L^1(B(1/6)))} \leq C(d, \alpha) \cdot M^{1-d} \quad (90)$$

for any  $M \geq 2^k$ . (Recall  $k = -3$ .)



To prove the above third claim (90), take a convolution first with  $\nabla^d[(h^\alpha)_M]$  into the equation (18). Then we have

$$\begin{aligned} & (\nabla^d[(h^\alpha)_M] * u)_t + (\nabla^d[(h^\alpha)_M] * ((w \cdot \nabla)u)) \\ & + (\nabla^d[(h^\alpha)_M] * \nabla P) - (\nabla^d[(h^\alpha)_M] * \Delta u) = 0 \end{aligned}$$

so that

$$\begin{aligned} & (\nabla^{d-1}[(h^\alpha)_M] * \nabla u)_t + (\nabla^d[(h^\alpha)_M] * ((w \cdot \nabla)u)) \\ & + (\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P) - \Delta(\nabla^{d-1}[(h^\alpha)_M] * \nabla u) = 0. \end{aligned}$$

Define a cut-off  $\Phi(t, x)$  by

$$\begin{aligned} 0 & \leq \Phi(x) \leq 1 \quad , \quad \text{supp}(\Phi) \subset (-4, 0) \times B(2) \\ \Phi(t, x) & = 1 \text{ for } (t, x) \in (-(1/6)^2, 0) \times B((1/6)). \end{aligned}$$

Multiply  $\Phi(t, x) \frac{(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t, x)}{|\nabla^{d-1}[(h^\alpha)_M] * \nabla u(t, x)|}$ , then integrate in  $x$ :

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \Phi(t, x) |(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t, x)| dx \\ & \leq \int_{\mathbb{R}^3} (|\partial_t \Phi(t, x)| + |\Delta \Phi(t, x)|) |(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t, x)| dx \\ & \quad + \int_{\mathbb{R}^3} |\Phi(t, x)| |(\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P)| dx \\ & \quad + \int_{\mathbb{R}^3} |\Phi(t, x)| |\nabla^d[(h^\alpha)_M] * ((w \cdot \nabla)u)| dx. \end{aligned}$$

Then integrating on  $[-4, t]$  for any  $t \in [-(1/6), 0]$  gives

$$\begin{aligned} & \| (h^\alpha)_M * \nabla^d u \|_{L^\infty(-(1/6)^2, 0; L^1(B(1/6)))} \\ & = \| \nabla^{d-1}[(h^\alpha)_M] * \nabla u \|_{L^\infty(-(1/6)^2, 0; L^1(B(1/6)))} \\ & \leq C \left( \| \nabla^{d-1}[(h^\alpha)_M] * \nabla u \|_{L^1(-4, 0; L^1(B(2)))} \right. \\ & \quad + \| \nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P \|_{L^1(-4, 0; L^1(B(2)))} \\ & \quad \left. + \| \nabla^d[(h^\alpha)_M] * ((w \cdot \nabla)u) \|_{L^1(-4, 0; L^1(B(2)))} \right) \\ & = (I) + (II) + (III). \end{aligned}$$

For (I), we use simple observations  $\nabla^m[(f)_\delta] = \delta^{-m} \cdot (\nabla^m f)_\delta$  and  $|(f)_\delta * \nabla u|(x) \leq C_f \cdot \mathcal{M}(|\nabla u|)(x)$  for any  $f \in C_0^\infty(\mathbb{R}^3)$  so that

$$\begin{aligned} & |(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t, x)| = M^{-(d-1)} \cdot |(\nabla^{d-1} h^\alpha)_M * \nabla u(t, x)| \\ & \leq C \cdot M^{-(d-1)} \cdot \mathcal{M}(|\nabla u|)(t, x) \end{aligned}$$

for any  $0 < M < \infty$  so that

$$\begin{aligned} (I) &= \|(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)\|_{L^1(-4,0;L^1(B(2)))} \\ &\leq C \cdot M^{-(d-1)} \cdot \|\mathcal{M}(|\nabla u|)\|_{L^1(-4,0;L^1(B(2)))} \\ &\leq C \cdot M^{-(d-1)} \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^2(B(2)))} \leq C \cdot M^{1-d} \end{aligned}$$

for any  $0 < M < \infty$ .

For (II), we use our global information about pressure in (23) thanks to the property of the Hardy space (10):

$$\begin{aligned} (II) &= \|\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P\|_{L^1(-4,0;L^1(B(2)))} \\ &= M^{-(d-1)} \cdot \|(\nabla^{d-1} h^\alpha)_M * \nabla^2 P\|_{L^1(-4,0;L^1(B(2)))} \\ &\leq M^{-(d-1)} \cdot \left\| \sup_{\delta > 0} |(\nabla^{d-1} h^\alpha)_\delta * \nabla^2 P| \right\|_{L^1(-4,0;L^1(B(2)))} \\ &\leq C \cdot M^{1-d} \end{aligned} \tag{91}$$

for any  $0 < M < \infty$ .

For (III), we use following useful facts **(1, ..., 5)**:

**1.** From  $\text{supp}((h^\alpha)_M) \subset B(2M)$ ,

$$\begin{aligned} &\|\nabla^d[(h^\alpha)_M] * ((w \cdot \nabla)u)(t, \cdot)\|_{L^1(B(2))} \\ &\leq \int_{B(2)} \int_{\mathbb{R}^3} \left| ((w \cdot \nabla)u)(t, y) \cdot (\nabla^d[(h^\alpha)_M])(x - y) \right| dy dx \\ &\leq \int_{B(2M+2)} \left| ((w \cdot \nabla)u)(t, y) \right| \cdot \left[ \int_{B(2)} |(\nabla^d[(h^\alpha)_M])(x - y)| dx \right] dy \\ &\leq C \|\nabla^d[(h^\alpha)_M]\|_{L^\infty(\mathbb{R}^3)} \cdot \|((w \cdot \nabla)u)(t, \cdot)\|_{L^1(B(2M+2))} \\ &\leq C \cdot \frac{1}{M^{3+d}} \cdot \|((w \cdot \nabla)u)(t, \cdot)\|_{L^1(B(2M+2))} \\ &\leq C \cdot \frac{1}{M^{3+d}} \cdot \|w(t, \cdot)\|_{L^{q'}(B(2M+2))} \cdot \|\nabla u(t, \cdot)\|_{L^q(B(2M+2))} \end{aligned}$$

where  $q = 12/(\alpha + 6)$  and  $1/q + 1/q' = 1$ .

Note: Because  $0 < \alpha < 2$ , we know  $12/8 < q < 2$ .

**2.**  $\|w(t, \cdot)\|_{L^q(B(2M+2))}$

$$\begin{aligned} &\leq C M^{1+\frac{3}{q}} \cdot \left( \|\mathcal{M}(|\nabla w|^q)(t, \cdot)\|_{L^1(B(1))}^{1/q} + \|\nabla w(t, \cdot)\|_{L^1(B(2))} \right) \\ &\leq C M^{1+\frac{3}{q}} \cdot \left( \|\mathcal{M}(|\mathcal{M}(|\nabla u|)^q)(t, \cdot)\|_{L^1(B(1))}^{1/q} + \|\mathcal{M}(|\nabla u|)(t, \cdot)\|_{L^1(B(2))} \right) \end{aligned}$$

for any  $M \geq 2^k$ .

For the first inequality, we used the lemma 4.4 and for the second one, we used the fact  $|\nabla w(t, x)| = |(\nabla u * \phi_r)(t, x)| \leq C|\mathcal{M}(|\nabla u|)(t, x)|$  where  $C$  is independent of  $0 \leq r < \infty$ . (For  $r > 0$ , it follows definitions of the convolution and the Maximal function while for  $r = 0$ , it follows the Lebesgue differentiation theorem with continuity of  $\nabla u$ .) So, for any  $M \geq 2^k$ , from (23),

$$\begin{aligned} & \|w\|_{L^2(-4,0;L^q(B(2M+2)))} \\ & \leq CM^{1+\frac{3}{q}} \left( \|\mathcal{M}(|\mathcal{M}(|\nabla u|)^q)\|_{L^1_x(B(1))}^{1/q} \|_{L^2_t(-4,0)} \right. \\ & \quad \left. + \|\mathcal{M}(|\nabla u|)\|_{L^1_x(B(2))} \|_{L^2_t(-4,0)} \right) \\ & \leq CM^{1+\frac{3}{q}} \left( \|\mathcal{M}(|\mathcal{M}(|\nabla u|)^q)\|_{L^{2/q}(-4,0;L^1(B(2)))}^{1/q} \right. \\ & \quad \left. + \|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^1(B(2)))} \right) \\ & \leq CM^{1+\frac{3}{q}}. \end{aligned}$$

Before stating the third fact, we needs the following two observations:

From standard Sobolev-Poincare inequality on balls (e.g. see Saloff-Coste [32]), we have  $C$  such that

$$\|f - \bar{f}\|_{L^{3q/(3-q)}(B(M))} \leq C \cdot \|\nabla f\|_{L^q(B(M))} \quad (92)$$

for any  $0 < M < \infty$  and for any  $f$  whose derivatives are in  $L^q_{loc}(\mathbb{R}^3)$  where  $\bar{f} = \int_B f dx / |B|$  is the mean value on  $B$ . Note that  $C$  is independent of  $M$ .

On the other hand, once we fix  $M_0 > 0$ , then there exist  $C = C(M_0)$  with the following property:

For any  $p$  with  $1 \leq p < \infty$ , for any  $M \geq M_0$  and for any  $f \in L^p_{loc}(\mathbb{R}^3)$ , we have

$$\|f\|_{L^p(B(M))} \leq CM^{\frac{3}{p}} \cdot \|\mathcal{M}(|f|^p)\|_{L^1(B(2))}^{1/p} \quad (93)$$

To prove (93), it is enough to show that

$$\|g\|_{L^1(B(M))} \leq CM^3 \cdot \|\mathcal{M}(g)\|_{L^1(B(2))}$$

For any  $z \in B(2)$ ,

$$\begin{aligned} \int_{B(M)} |g(x)| dx &= \frac{(M+2)^3}{(M+2)^3} \cdot \int_{B(M+2)} |g(z+x)| dx \\ &\leq (M+2)^3 \mathcal{M}(g)(z) \leq C_{M_0} M^3 \mathcal{M}(g)(z) \end{aligned}$$

Then we take integral on  $z \in B(2)$ .

Now we states the third fact.

$$\begin{aligned}
\mathbf{3.} \quad & \|w(t, \cdot)\|_{L^{3q/(3-q)}(B(2M+2))} \\
& \leq C \cdot \|\nabla w(t, \cdot)\|_{L^q(B(2M+2))} + \|\bar{w}(t, \cdot)\|_{L^{3q/(3-q)}(B(2M+2))} \\
& \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla w|^q)(t, \cdot)\|_{L^1(B(2))}^{1/q} \\
& \quad + CM^{-3} \|w(t, \cdot)\|_{L^1(B(2M+2))} \cdot CM^{3 \cdot \frac{3-q}{3q}} \\
& \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|^q)(t, \cdot)\|_{L^1(B(2))}^{1/q} \\
& \quad + CM^{\frac{3}{q}-4} \|w(t, \cdot)\|_{L^1(B(2M+2))} \\
& \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|^q)(t, \cdot)\|_{L^1(B(2))}^{1/q} \\
& \quad + CM^{\frac{3}{q}-4} CM^{1+\frac{3}{1}} \cdot \left( \|\mathcal{M}(|\nabla w|^1)(t, \cdot)\|_{L^1(B(1))}^{1/1} + \|\nabla w(t, \cdot)\|_{L^1(B(2))} \right) \\
& \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|^q)(t, \cdot)\|_{L^1(B(2))}^{1/q} \\
& \quad + CM^{\frac{3}{q}} \left( \|\mathcal{M}(|\mathcal{M}(|\nabla u|^q)(t, \cdot)\|_{L^1(B(1))} + \|\mathcal{M}(|\nabla u|)(t, \cdot)\|_{L^1(B(2))} \right)
\end{aligned}$$

we used (92) for the first inequality, (93) and definition of mean value for the second one and  $|\nabla w(t, x)| \leq C|\mathcal{M}(|\nabla u|)(t, x)|$  and the lemma 4.4 for fourth and fifth ones respectively. So, by taking  $L^2$ -norm on time  $[-4, 0]$  with (23),

$$\|w\|_{L^2(-4,0;L^{\frac{3q}{3-q}}(B(2M+2)))} \leq CM^{\frac{3}{q}}$$

for any  $M \geq 2^k$ .

$$\mathbf{4.} \quad \|w(t, \cdot)\|_{L^{q'}(B(2M+2))} \leq \|w(t, \cdot)\|_{L^q(B(2M+2))}^\theta \cdot \|w(t, \cdot)\|_{L^{3q/(3-q)}(B(2M+2))}^{1-\theta}$$

where  $q' = q/(q-1)$  and  $\theta = (4q-6)/q$ .

Note: Because  $12/8 < q < 2$ , we have  $0 < \theta < 1$ . So, for any  $M \geq 2^k$ ,

$$\begin{aligned}
& \|w\|_{L^2(-4,0;L^{q'}(B(2M+2)))} \\
& \leq \|w\|_{L^2(-4,0;L^q(B(2M+2)))}^\theta \cdot \|w\|_{L^2(-4,0;L^{3q/(3-q)}(B(2M+2)))}^{1-\theta} \\
& \leq C \cdot (M^{1+(3/q)})^\theta (M^{3/q})^{1-\theta} = C \cdot M^{4-\frac{3}{q}}.
\end{aligned}$$

**5.** From (93), for any  $M \geq 2^k$ ,

$$\|\nabla u(t, \cdot)\|_{L^q(B(2M+2))} \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla u|^q)(t, \cdot)\|_{L^1(B(2))}^{1/q}.$$

So, for any  $M \geq 2^k$ , from (23),

$$\begin{aligned}
\|\nabla u\|_{L^2(-4,0;L^q(B(2M+2)))} & \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla u|^q)\|_{L_x^1(B(2))}^{1/q} \|_{L_t^2(-4,0)} \\
& \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla u|^q)\|_{L^{2/q}(-4,0;L^1(B(2)))}^{1/q} \leq C \cdot M^{3/q}.
\end{aligned}$$

Using above five results **(1, ..., 5)** all together, we have for any  $M \geq 2^k$ ,

$$\begin{aligned} (III) &\leq C \cdot \frac{1}{M^{3+d}} \cdot \|w\|_{L^2(-4,0;L^{q'}(B(2M+2)))} \|\nabla u\|_{L^2(-4,0;L^q(B(2M+2)))} \\ &\leq C \cdot \frac{1}{M^{3+d}} \cdot M^{4-(3/q)} \cdot M^{3/q} = C \cdot M^{1-d} \end{aligned}$$

which proved the above third claim (90).

Finally we combine three claims (88), (89) and (90):

$$\begin{aligned} &\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))} \\ &\leq \|C \left(1 + \left| \int_{|y| \geq (1/6)} \frac{\nabla^d u(\cdot, \cdot, x-y)}{|y|^{3+\alpha}} dy \right| \right)\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))} \\ &\leq C + C \sum_{j=k}^{\infty} \left(\frac{1}{2^\alpha}\right)^j \cdot \|((h^\alpha)_{2j} * \nabla^d u)(\cdot, \cdot, x)\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))} \\ &\leq C + C \sum_{j=k}^{\infty} \left(\frac{1}{2^\alpha}\right)^j \cdot (2^j)^{1-d} \leq C + C \sum_{j=k}^{\infty} \left(\frac{1}{2^{d+\alpha-1}}\right)^j \leq C \end{aligned}$$

because  $d + \alpha - 1 > 0$  from  $d \geq 1$  and  $\alpha > 0$ .

By the exact same way, we can also prove that

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^m u\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))} \leq C$$

for  $m = d+1, \dots, d+4$ . By repeated uses of Sobolev's inequality,

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u\|_{L^\infty(-(1/6)^2,0;L^\infty(B((1/6))))} \leq C(d, \alpha)$$

and it finishes this proof of the proposition 2.2.  $\square$

## 5 Proof of the main theorem 1.1

We begin this section by presenting one small lemma about pivot quantities. After that, the subsection 5.2 covers the part (II) for  $\alpha = 0$  while the subsection 5.3 does the part (II) for  $0 < \alpha < 2$ . Finally the part (I) for  $0 \leq \alpha < 2$  follows in the subsection 5.4.

### 5.1 $L^1$ Pivot quantities

The following lemma says that  $L^1$  space-time norm of our pivot quantities can be controlled by  $L^2$  space norm of the initial data. These things have the best scaling like  $|\nabla u|^2$  and  $|\nabla^2 P|$  among all other *a priori* quantities from  $L^2$  initial data (also see (4)).

**Lemma 5.1.** *There exist constant  $C > 0$  and  $C_{d,\alpha}$  for integer  $d \geq 1$  and real  $\alpha \in (0, 2)$  with the following property:*

*If  $(u, P)$  is a solution of (Problem I- $n$ ) for some  $1 \leq n \leq \infty$ , then we have*

$$\int_0^\infty \int_{\mathbb{R}^3} (|\nabla u(t, x)|^2 + |\nabla^2 P(t, x)| + |\mathcal{M}(|\nabla u|)(t, x)|^2) dx dt \leq C \|u_0\|_{L^2(\mathbb{R}^3)}^2$$

*and*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} \left( |\mathcal{M}(\mathcal{M}(|\nabla u|))|^2 + |\mathcal{M}(|\nabla u|^q)|^{2/q} + |\mathcal{M}(|\mathcal{M}(|\nabla u|)^q)|^{2/q} \right. \\ & \quad \left. + \sum_{m=d}^{d+4} \sup_{\delta > 0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P|) \right) dx dt \leq C_{d,\alpha} \|u_0\|_{L^2(\mathbb{R}^3)}^2 \end{aligned}$$

*for any integer  $d \geq 1$  and any real  $\alpha \in (0, 2)$  where  $q = q(\alpha)$  is defined by  $12/(\alpha + 6)$ .*

*Remark 5.1.* The definitions of  $h^\alpha$  and  $(\nabla^{m-1} h^\alpha)_\delta$  can be found around (12).

*Remark 5.2.* In the following proof, we will see that every quantity in the left hand sides of the above two estimates can be controlled by dissipation of energy  $\|\nabla u\|_{L^2((0,\infty) \times \mathbb{R}^3)}^2$  only. It explains the latter part of the remark 1.2.

*Proof.* From (17),

$$\|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \leq \|u_0 * \phi_{\frac{1}{n}}\|_{L^2(\mathbb{R}^3)}^2 \leq \|u_0\|_{L^2(\mathbb{R}^3)}^2.$$

For the pressure term, we use boundedness of the Riesz transform on Hardy space  $\mathcal{H}$  and compensated compactness result in Coifman, Lions, Meyer and Semmes [11]:

$$\begin{aligned} \|\nabla^2 P\|_{L^1(0,\infty;L^1(\mathbb{R}^3))} & \leq \|\nabla^2 P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \leq C \|\Delta P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \\ & = \|\operatorname{div} \operatorname{div} \left( (u * \phi_{1/n}) \otimes u \right)\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \\ & \leq C \cdot \|\nabla(u * \phi_{1/n})\|_{L^2(0,\infty;L^2(\mathbb{R}^3))} \|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ & \leq C \cdot \|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \leq C \|u_0\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \tag{94}$$

For Maximal functions,

$$\begin{aligned} \|\mathcal{M}(\mathcal{M}(|\nabla u|))\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 & \leq C \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \\ & \leq C \cdot \|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \\ & \leq C \cdot \|u_0\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

Let  $d \geq 1$  and  $0 < \alpha < 2$  and take  $q = 12/(\alpha + 6)$ . From  $1 < (2/q) < (4/3)$ ,

$$\begin{aligned} \|\mathcal{M}(|\nabla u|^q)\|_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))}^{2/q} & \leq C \cdot \|\nabla u\|_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))}^{2/q} \\ & = C \cdot \|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \\ & \leq C \cdot \|u_0\|_{L^2(\mathbb{R}^3)}^2 \end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{M}(|\mathcal{M}(|\nabla u)|^q)\|_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))}^{2/q} &\leq C \cdot \|\mathcal{M}(|\nabla u|)^q\|_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))}^{2/q} \\
&\leq C \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \\
&\leq C \cdot \|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}^2 \\
&\leq C \cdot \|u_0\|_{L^2(\mathbb{R}^3)}^2
\end{aligned}$$

where  $C$  depends only on  $\alpha$ .

Thanks to the property of Hardy space (10) with (94), we have

$$\begin{aligned}
\sum_{m=d}^{d+4} \left\| \sup_{\delta>0} (|\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P| \right\|_{L^1(0,\infty;L^1(\mathbb{R}^3))} &\leq \sum_{m=d}^{d+4} C \|\nabla^2 P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \\
&\leq C \|u_0\|_{L^2(\mathbb{R}^3)}^2
\end{aligned}$$

where the above  $C$  depends only on  $d$  and  $\alpha$ .  $\square$

We are ready to prove the main theorem 1.1.

*Remark 5.3.* In the following subsections 5.2 and 5.3, we consider solutions of (Problem I-n) for positive integers  $n$ . However it will be clear that every computation in these subsections can also be verified for the case  $n = \infty$  once we assume that the smooth solution  $u$  of the Navier-Stokes exists. This  $n = \infty$  case (the original Navier-Stokes) will be covered in the subsection 5.4.

We focus on the  $\alpha = 0$  case of the part (II) first.

## 5.2 Proof of theorem 1.1 part (II) for $\alpha = 0$ case

*Proof of proposition 1.1 part (II) for the  $\alpha = 0$  case.*

Let any  $u_0$  of (2) be given. From the Leray's construction, there exists the  $C^\infty$  solution sequence  $\{u_n\}_{n=1}^\infty$  of (Problem I-n) on  $(0, \infty)$  with corresponding pressures  $\{P_n\}_{n=1}^\infty$ . From now on, our goal is to make an estimate for  $\nabla^d u_n$  which is uniform in  $n$ .

For each  $n$ ,  $\epsilon > 0$ ,  $t > 0$  and  $x \in \mathbb{R}^3$ , define a new flow  $X_{n,\epsilon}(\cdot, t, x)$  by solving

$$\begin{aligned}
\frac{\partial X_{n,\epsilon}}{\partial s}(s, t, x) &= u_n * \phi_{\frac{1}{n}} * \phi_\epsilon(s, X_{n,\epsilon}(s, t, x)) \quad \text{for } s \in [0, t], \\
X_{n,\epsilon}(t, t, x) &= x.
\end{aligned}$$

For convenience, we define  $F_n(t, x)$  and  $g_n(t)$ .

$$F_n(t, x) = (|\nabla u_n|^2 + |\nabla^2 P_n| + |\mathcal{M}(\nabla u_n)|^2)(t, x), \quad g_n(t) = \int_{\mathbb{R}^3} F_n(t, x) dx.$$

We define for  $n, t > 0$  and  $0 < 4\epsilon^2 \leq t$

$$\Omega_{n,\epsilon,t} = \{x \in \mathbb{R}^3 \mid \frac{1}{\epsilon} \int_{t-4\epsilon^2}^t \int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s, t, x) + y) dy ds \leq \bar{\eta}\}$$

where  $\bar{\eta}$  comes from the proposition 2.2. We measure size of  $(\Omega_{n,\epsilon,t})^C$ :

$$\begin{aligned} |(\Omega_{n,\epsilon,t})^C| &= |\{x \in \mathbb{R}^3 \mid \frac{1}{\epsilon} \int_{t-4\epsilon^2}^t \int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s, t, x) + y) dy ds > \bar{\eta}\}| \\ &\leq \frac{1}{\bar{\eta}} \int_{\mathbb{R}^3} \left( \frac{1}{\epsilon} \int_{t-4\epsilon^2}^t \int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s, t, x) + y) dy ds \right) dx \\ &= \frac{1}{\bar{\eta}\epsilon} \left( \int_{B(2\epsilon)} \int_{-4\epsilon^2}^0 \int_{\mathbb{R}^3} F_n(t+s, X_{n,\epsilon}(t+s, t, x) + y) dx ds dy \right) \\ &= \frac{1}{\bar{\eta}\epsilon} \left( \int_{B(2\epsilon)} \int_{-4\epsilon^2}^0 \int_{\mathbb{R}^3} F_n(t+s, z+y) dz ds dy \right) \\ &\leq \frac{1}{\bar{\eta}\epsilon} \left( \int_{B(2\epsilon)} 1 dy \right) \left( \int_{-4\epsilon^2}^0 \int_{\mathbb{R}^3} F_n(t+s, \bar{z}) d\bar{z} ds \right) \\ &\leq \frac{C\epsilon^2}{\bar{\eta}} \left( \int_{-4\epsilon^2}^0 \int_{\mathbb{R}^3} F_n(t+s, \bar{z}) d\bar{z} ds \right) \\ &\leq C \frac{\epsilon^4}{\bar{\eta}} \left( \frac{1}{4\epsilon^2} \int_{-4\epsilon^2}^0 g_n(t+s) ds \right) \leq \epsilon^4 \mathcal{M}^{(t)} \left( \frac{C}{\bar{\eta}} g_n \cdot \mathbf{1}_{(0,\infty)} \right)(t) = \epsilon^4 \tilde{g}_n(t) \end{aligned} \tag{95}$$

where  $\tilde{g}_n = \mathcal{M}^{(t)} \left( \frac{C}{\bar{\eta}} g_n \cdot \mathbf{1}_{(0,\infty)} \right)$  and  $\mathcal{M}^{(t)}$  is the Maximal function in  $\mathbb{R}^1$ . For the third inequality, we used the fact that  $X_{n,\epsilon}(\cdot, t, x)$  is incompressible. From the fact that the Maximal operator is bounded from  $L^1$  to  $L^{1,\infty}$  together with the lemma 5.1,  $\|\tilde{g}_n(\cdot)\|_{L^{1,\infty}(0,\infty)} \leq \frac{C}{\bar{\eta}} \|g_n(\cdot)\|_{L^1(0,\infty)} \leq \frac{C}{\bar{\eta}} \|u_0\|_{L^2(\mathbb{R}^3)}^2$ .

Now we fix  $n, t, \epsilon$  and  $x$  with  $n \geq 1$ ,  $0 < t < \infty$ ,  $0 < 4\epsilon^2 \leq t$  and  $x \in \Omega_{n,\epsilon,t}$ . We define  $v, Q$  on  $(-4, \infty) \times \mathbb{R}^3$  by using the Galilean invariance:

$$\begin{aligned} v(s, y) &= \epsilon u_n(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y) \\ &\quad - \epsilon(u_n * \phi_\epsilon)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x)) \\ Q(s, y) &= \epsilon^2 P_n(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y) \\ &\quad + \epsilon y \partial_s [(u_n * \phi_\epsilon)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x))]. \end{aligned} \tag{96}$$

*Remark 5.4.* This specially designed  $\epsilon$ -scaling will give the mean zero property to both the velocity and the advection velocity of the resulting equation (97).

Let us denote  $\square$  and  $\diamond$  by  $\square = (t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y)$  and  $\diamond =$



$(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x))$ , respectively. Then the chain rule gives us

$$\begin{aligned} \partial_s v(s, y) &= \epsilon^3 \partial_t(u_n)(\square) + \epsilon^3 ((u_n * \phi_{\frac{1}{n}} * \phi_\epsilon)(\diamond) \cdot \nabla) u_n(\square) - \epsilon \partial_s [(u_n * \phi_\epsilon)(\diamond)], \\ (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, y) &= \epsilon (u_n * \phi_{\frac{1}{n}})(\square) - \epsilon (u_n * \phi_\epsilon)(\diamond), \\ \int_{\mathbb{R}^3} (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, z) \phi(z) dz &= \epsilon (u_n * \phi_{\frac{1}{n}} * \phi_\epsilon)(\diamond) - \epsilon (u_n * \phi_\epsilon)(\diamond), \\ \left( \left( (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, y) - \int_{\mathbb{R}^3} (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, z) \phi(z) dz \right) \cdot \nabla \right) v(s, y) &= \\ \epsilon^3 \left( (u_n * \phi_{\frac{1}{n}})(\square) \cdot \nabla \right) u_n(\square) - \epsilon^3 \left( (u_n * \phi_{\frac{1}{n}} * \phi_\epsilon)(\diamond) \cdot \nabla \right) u_n(\square), \\ -\Delta_y v(s, y) &= -\epsilon^3 \Delta_y u_n(\square) \text{ and} \\ \nabla_y Q(s, y) &= \epsilon^3 \nabla P_n(\square) + \epsilon \partial_s [(u_n * \phi_\epsilon)(\diamond)]. \end{aligned}$$

Thus, for  $(s, y) \in (-4, \infty) \times \mathbb{R}^3$ ,

$$\left[ \partial_s v + \left( \left( (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, y) - \int_{\mathbb{R}^3} (v *_{y, \phi_{\frac{1}{n\epsilon}}})(s, z) \phi(z) dz \right) \cdot \nabla \right) v + \nabla Q - \Delta v \right](s, y) = 0. \quad (97)$$

As a result,  $(v(\cdot_s, \cdot_y), Q(\cdot_s, \cdot_y))$  is a solution of (Problem II- $\frac{1}{n\epsilon}$ ).

From definition of the Maximal function, we can verify that  $|\mathcal{M}(\nabla v)|^2$  behaves like  $|\nabla v|^2$  under the scaling in the following sense:

$$\begin{aligned} \mathcal{M}(\nabla v)(s, y) &= \sup_{M>0} \frac{C}{M^3} \int_{B(M)} \epsilon^2 (\nabla u_n)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon(y + z)) dz \\ &= \sup_{\epsilon M > 0} \frac{C}{\epsilon^3 M^3} \int_{B(\epsilon M)} \epsilon^2 (\nabla u_n)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y + \bar{z}) d\bar{z} \\ &= \epsilon^2 \mathcal{M}(\nabla u_n)(\square) \end{aligned} \quad (98)$$

As a result,

$$\begin{aligned} &\int_{-4}^0 \int_{B(2)} (|\nabla v(s, y)|^2 + |\nabla^2 Q(s, y)| + |\mathcal{M}(\nabla v)(s, y)|^2) dy ds \\ &= \epsilon^4 \int_{-4}^0 \int_{B(2)} \left[ |\nabla u_n|^2 + |\nabla^2 P_n| + |\mathcal{M}(\nabla u_n)|^2 \right](\square) dy ds \\ &= \epsilon^4 \int_{-4}^0 \int_{B(2)} F_n(\square) dy ds \\ &= \epsilon^{-1} \int_{t-4\epsilon^2}^t \int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s, t, x) + y) dy ds \leq \bar{\eta} \end{aligned}$$

where the first equality comes from the definition of  $(v, Q)$  and the second one follows the change of variable  $(t + \epsilon^2 s, \epsilon y) \rightarrow (s, y)$ . Moreover, it satisfies

$$\int_{\mathbb{R}^3} \phi(z) v(s, z) dz = 0, \quad -4 < s < 0. \quad (99)$$

So  $(v, Q)$  satisfies all conditions (21, 22) in the proposition 2.2 with  $r = 1/(n\epsilon) \in [0, \infty)$ .

The conclusion of the proposition 2.2 implies that if  $x \in \Omega_{n,\epsilon,t}$  for some  $n, t$  and  $\epsilon$  such that  $4\epsilon^2 \leq t$  then  $|\nabla^d v(0, 0)| \leq C_d$ . As a result, using  $\nabla^d v(0, 0) = \epsilon^{d+1} \nabla^d u_n(t, x)$  for any integer  $d \geq 1$ , we have

$$|\{x \in \mathbb{R}^3 \mid |\nabla^d u_n(t, x)| > \frac{C_d}{\epsilon^{d+1}}\}| \leq |\Omega_{n,\epsilon,t}^C| \leq \epsilon^4 \cdot \tilde{g}_n(t).$$

Let  $K$  be any open bounded subset in  $\mathbb{R}^3$ . Also define  $p = 4/(d+1)$ . Then for any  $t > 0$ ,

$$\beta^p \cdot \left| \{x \in K : |(\nabla^d u_n)(t, x)| > \beta\} \right| \leq \begin{cases} \beta^p \cdot |K|, & \text{if } \beta \leq C \cdot t^{-2/p} \\ C \cdot \tilde{g}_n(t), & \text{if } \beta > C \cdot t^{-2/p}. \end{cases}$$

Thus,

$$\|(\nabla^d u_n)(t, \cdot)\|_{L^{p,\infty}(K)}^p \leq C \cdot \max(\tilde{g}_n(t), \frac{|K|}{t^2})$$

We pick any  $t_0 > 0$ . If we take  $L^{1,\infty}(t_0, T)$ -norm to the above inequality, then we obtain

$$\begin{aligned} \|\nabla^d u_n\|_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))}^p &\leq C \left( \|\tilde{g}_n\|_{L^{1,\infty}(0,\infty)} + |K| \cdot \left\| \frac{1}{|\cdot|^2} \right\|_{L^{1,\infty}(t_0,\infty)} \right) \\ &\leq C \left( \|u_0\|_{L^2(\mathbb{R}^3)}^2 + \frac{|K|}{t_0} \right) \end{aligned} \tag{100}$$

where  $C$  depends only on  $d \geq 1$ .

We observe that the above estimate is uniform in  $n$ . It is well known that both  $\nabla u$  and  $\nabla^2 u$  are locally integrable functions for any suitable weak solution  $u$  which can be obtained by a limiting argument of  $u_n$  (e.g. see Lions [29]). Thus, the above estimates (100) holds even for  $u$  with  $d = 1, 2$ .

*Remark 5.5.* In fact, for the case  $d = 1$ , the above estimate says  $\nabla u \in L_{loc}^{2,\infty}$ , which is useless because we know a better estimate  $\nabla u \in L^2$ .

*Remark 5.6.* For  $d \geq 3$ , the above estimate (100) does not give us any direct information about higher derivatives  $\nabla^d u$  of a weak solution  $u$  because full regularity of weak solutions is still open, so  $\nabla^d u$  may not be locally integrable for  $d \geq 3$ . Instead, the only thing we can say is that, for  $d \geq 3$ , higher derivatives  $\nabla^d u_n$  of a Leray's approximation  $u_n$  have  $L_{loc}^{4/(d+1),\infty}$  bounds which are uniform in  $n \geq 1$ .

□

From now on, we will prove the  $0 < \alpha < 2$  case of the part (II).

### 5.3 Proof of theorem 1.1 part (II) for $0 < \alpha < 2$ case

*Proof of proposition 1.1 part (II) for the  $0 < \alpha < 2$  case.*

We fix  $d \geq 1$  and  $0 < \alpha < 2$ . Then, for any positive integer  $n$ , any  $t > 0$  and  $x \in \mathbb{R}^3$ , we denote  $F_n(t, x)$  in this time by:

$$\begin{aligned} F_n(t, x) = & \left( |\nabla u_n(t, x)|^2 + |\nabla^2 P_n(t, x)| + |\mathcal{M}(\nabla u_n)(t, x)|^2 \right. \\ & + |\mathcal{M}(\mathcal{M}(|\nabla u_n|))|^2 + (\mathcal{M}(|\mathcal{M}(|\nabla u_n|)^q))^{2/q} \\ & \left. + |\mathcal{M}(|\nabla u_n|^q)|^{2/q} + \sum_{m=d}^{d+4} \sup_{\delta>0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P|) \right). \end{aligned}$$

We use the same definitions for  $g_n$ ,  $\tilde{g}_n$ ,  $X_{n,\epsilon}$  and  $\Omega_{n,\epsilon,t}$  of the previous section 5.2 for the case  $\alpha = 0$ . Note that they depend on  $d$  and  $\alpha$ , and we have  $\|\tilde{g}_n\|_{L^{1,\infty}(0,\infty)} \leq \frac{C_{d,\alpha}}{\tilde{\eta}} \cdot \|u_0\|_{L^2(\mathbb{R}^3)}^2$  from the lemma 5.1.

Now we pick any  $x \in \Omega_{n,\epsilon,t}$  and any  $\epsilon$  such that  $4\epsilon^2 \leq t$ , and define  $v$  and  $Q$  as the previous section 5.2 (see (96)).

In order to follow the previous subsection 5.2, only thing which remains is to verify if every quantity in  $F_n(t, x)$  has the same scaling with  $|\nabla v|^2$  after the transform (96). For Maximal of Maximal functions,

$$\begin{aligned} & \mathcal{M}(\mathcal{M}(|\nabla v|))(s, y) \\ &= \sup_{M>0} \frac{C}{M^3} \int_{B(M)} \mathcal{M}(|\nabla v|)(s, y+z) dz \\ &= \sup_{M>0} \frac{C}{M^3} \int_{B(M)} \epsilon^2 \mathcal{M}(|\nabla u_n|)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon(y+z)) dz \\ &= \epsilon^2 \mathcal{M}(\mathcal{M}(|\nabla u_n|))(\square). \end{aligned}$$

where  $\square = (t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y)$  and we used the idea of (98) for second and third equalities. Likewise,  $\mathcal{M}(|\nabla v|^q)(s, y) = \epsilon^{2q} \cdot \mathcal{M}(|\nabla u_n|^q)(\square)$  and  $\mathcal{M}(|\mathcal{M}(|\nabla v|)^q)(s, y) = \epsilon^{2q} \cdot \mathcal{M}(|\mathcal{M}(|\nabla u_n|)^q)(\square)$ .

Also, we have for any function  $\mathcal{G} \in C_0^\infty$ ,

$$\begin{aligned} \sup_{\delta>0} (|\mathcal{G}_\delta * \nabla^2 Q|)(s, y) &= \sup_{\delta>0} \left| \int_{\mathbb{R}^3} \frac{1}{\delta^3} \mathcal{G}\left(\frac{z}{\delta}\right) \cdot (\nabla^2 Q)(s, y-z) dz \right| \\ &= \sup_{\delta>0} \left| \int_{\mathbb{R}^3} \frac{\epsilon^4}{\delta^3} \mathcal{G}\left(\frac{z}{\delta}\right) \cdot (\nabla^2 P_n)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon(y-z)) dz \right| \\ &= \sup_{\delta>0} \left| \int_{\mathbb{R}^3} \frac{\epsilon^4}{\epsilon^3 \delta^3} \mathcal{G}\left(\frac{z}{\epsilon \delta}\right) \cdot (\nabla^2 P_n)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y - z) dz \right| \end{aligned}$$

$$\begin{aligned}
 &= \sup_{\epsilon\delta>0} \left| \int_{\mathbb{R}^3} \epsilon^4 \mathcal{G}_{\epsilon\delta}(z) \cdot (\nabla^2 P_n)(t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y - z) dz \right| \\
 &= \sup_{\epsilon\delta>0} \epsilon^4 \left| \left( \mathcal{G}_{\epsilon\delta} * (\nabla^2 P_n) \right) (t + \epsilon^2 s, X_{n,\epsilon}(t + \epsilon^2 s, t, x) + \epsilon y) \right| \\
 &= \epsilon^4 \sup_{\delta>0} \left| \mathcal{G}_\delta * (\nabla^2 P_n) \right|(\square).
 \end{aligned}$$

Thus by taking  $\mathcal{G} = (\nabla^{m-1} h^\alpha)$ , we have

$$\sup_{\delta>0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 Q|)(s, y) = \epsilon^4 \sup_{\delta>0} \left| (\nabla^{m-1} h^\alpha)_\delta * (\nabla^2 P_n) \right|(\square).$$

As a result, we have

$$\begin{aligned}
 &\int_{-4}^0 \int_{B(2)} \left[ |\nabla v|^2 + |\nabla^2 Q| + |\mathcal{M}(\nabla v)|^2 + \right. \\
 &\quad \left. + |\mathcal{M}(\mathcal{M}(|\nabla v|))|^2 + |\mathcal{M}(|\mathcal{M}(|\nabla v|)|^q)|^{q/2} \right. \\
 &\quad \left. + |\mathcal{M}(|\nabla v|^q)|^{2/q} + \sum_{m=d}^{d+4} \sup_{\delta>0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 Q|) \right] (s, y) dy ds \\
 &= \epsilon^4 \int_{-4}^0 \int_{B(2)} F_n(\square) dy ds \\
 &= \epsilon^{-1} \int_{t-4\epsilon^2}^t \int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s, t, x) + y) dy ds \leq \bar{\eta}.
 \end{aligned}$$

Then  $(v, Q)$  satisfies condition (23) as well as (21) and (22) of the proposition 2.2 with  $r = 1/(n\epsilon) \in [0, \infty)$ . In sum if  $x \in \Omega_{n,\epsilon,t}$  and if  $4\epsilon^2 \leq t$ , then

$$|(-\Delta)^{\alpha/2} \nabla^d v(0, 0)| \leq C_{d,\alpha}.$$

Because  $u_n$  is a smooth solution of (Problem I-n),  $(-\Delta)^{\alpha/2} \nabla^d u_n$  is not only a distribution but also a locally integrable function. Indeed, from a boot-strapping argument, it is easy to show that  $\nabla^d u_n(t)$  has a good behavior at infinity which is required in order to use the integral representation (14) pointwise. For example,  $(C^2 \cap W^{2,\infty})$  is enough (For a better approach, see Silvestre [37]). Also it can be easily verified that the resulting function  $(-\Delta)^{\alpha/2} [\nabla^d u_n(t, \cdot)](x)$  from the integral representation (14) satisfies the definition in the remark 1.1.

As a result, it makes sense to talk about pointwise values of  $(-\Delta)^{\alpha/2} \nabla^d u_n$ . Thus, from the simple observation: for any integer  $d \geq 1$  and any real  $0 < \alpha < 2$ ,

$$(-\Delta)^{\alpha/2} \nabla^d v(0, 0) = \epsilon^{d+\alpha+1} (-\Delta)^{\alpha/2} \nabla^d u_n(t, x),$$

we can deduce the following set inclusion:

$$\{x \in \mathbb{R}^3 \mid |(-\Delta)^{\frac{\alpha}{2}} \nabla^d u_n(t, x)| > \frac{C_{d,\alpha}}{\epsilon^{d+\alpha+1}}\} \subset \Omega_{n,\epsilon,t}^C. \quad (101)$$

Thus we have for any  $0 < t < \infty$  and for any  $0 < 4\epsilon^2 \leq t$

$$|\{x \in \mathbb{R}^3 \mid |(-\Delta)^{\frac{\alpha}{2}} \nabla^d u_n(t, x)| > \frac{C_{d,\alpha}}{\epsilon^{d+\alpha+1}}\}| \leq |\Omega_{n,\epsilon,t}^C| \leq \epsilon^4 \cdot \tilde{g}_n(t).$$

Define  $p = 4/(d + \alpha + 1)$ . Like we did for case  $\alpha = 0$ , we obtain

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u_n\|_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))}^p \leq C \left( \|u_0\|_{L^2(\mathbb{R}^3)}^2 + \frac{|K|}{t_0} \right)$$

for any integer  $n, d \geq 1$ , for any real  $\alpha \in (0, 2)$ , for any bounded open subset  $K$  of  $\mathbb{R}^3$  and for any  $t_0 \in (0, \infty)$  where  $C$  depends only on  $d$  and  $\alpha$ .

If we restrict further  $(d + \alpha) < 3$ , then  $p = \frac{4}{d+\alpha+1} > 1$ . This implies  $(-\Delta)^{\alpha/2} \nabla^d u_n \in L_{loc}^q((t_0, \infty) \times K)$  for every  $q$  between 1 and  $p$ , and the norm is uniformly bounded in  $n$ . Thus, from weak-compactness of  $L^q$  for  $q > 1$ , we conclude that if  $u$  is a suitable weak solution obtained by a limiting argument of  $u_n$ , then any higher derivatives  $(-\Delta)^{\alpha/2} \nabla^d u$ , which is defined in the remark 1.1, lie in  $L_{loc}^1$  as long as  $(d + \alpha) < 3$  with the same estimate

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u\|_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))}^p \leq C_{d,\alpha} \left( \|u_0\|_{L^2(\mathbb{R}^3)}^2 + \frac{|K|}{t_0} \right). \quad (102)$$

□

## 5.4 Proof of theorem 1.1 part (I)

*Proof of proposition 1.1 part (I).* Suppose that  $(u, P)$  is a smooth solution of the Navier-Stokes equations (1) on  $(0, T)$  with (2). Then it satisfies all conditions of (Problem I-n) for  $n = \infty$  on  $(0, T)$ . As we mentioned at the remark 5.3, we follow every steps in the subsections 5.2 and 5.3 except each last arguments which impose  $d < 3$  or  $(d + \alpha) < 3$ . Indeed, under the scaling (96), the resulting function  $(v, Q)$  is a solution for (Problem II-r) for  $r = 0$ .

Recall that  $u$  is smooth by assumption. As a result, we do NOT have any restriction like  $d < 3$  or  $(d + \alpha) < 3$  at this time because we do not need any limiting argument any more which requires a weak-compactness. Thus, we obtain (102) for any integer  $d \geq 1$ , for any real  $\alpha \in [0, 2)$  and for any  $t_0 \in (0, T)$ . It finishes the proof of the part (I) of the main theorem 1.1.

□

## A. Appendix: proofs of some technical lemmas

*proof for lemma 4.1.* Fix  $(n, a, b, p)$  such that  $n \geq 2$ ,  $0 < b < a < 1$  and  $1 < p < \infty$ . Let  $\alpha$  be any multi index such that  $|\alpha| = n$  and  $D^\alpha = \partial_{\alpha_1} \partial_{\alpha_2} D^\beta$  where  $\beta$  is a multi index with  $|\beta| = n - 2$ .

Observe that from  $\operatorname{div}(v_2) = 0$  and  $\operatorname{div}(v_1) = 0$ ,

$$\begin{aligned} -\Delta(D^\alpha P) &= \operatorname{div} \operatorname{div} D^\alpha(v_2 \otimes v_1) \\ &= D^\alpha \left( \sum_{ij} (\partial_j v_{2,i})(\partial_i v_{1,j}) \right) \\ &= \partial_{\alpha_1} \partial_{\alpha_2} H \end{aligned}$$

where  $H = D^\beta \left( \sum_{ij} (\partial_j v_{2,i})(\partial_i v_{1,j}) \right)$  and  $v_k = (v_{k,1}, v_{k,2}, v_{k,3})$  for  $k = 1, 2$ .  
Then for any  $(p_1, p_2)$  such that  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

$$\|H\|_{L^p(B(a))} \leq C \|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))}$$

where  $C$  is independent of choice of  $p_1$  and  $p_2$  and

$$\|H\|_{W^{1,\infty}(B(a))} \leq C \|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))}.$$

Fix a function  $\psi \in C^\infty(\mathbb{R}^3)$  satisfying:

$$\psi = 1 \quad \text{in } B(b + \frac{a-b}{3}), \quad \psi = 0 \quad \text{in } (B(b + \frac{2(a-b)}{3}))^C \text{ and } 0 \leq \psi \leq 1.$$

We decompose  $D^\alpha P$  by using  $\psi$ :

$$\begin{aligned} -\Delta(\psi D^\alpha P) &= -\psi \Delta D^\alpha P - 2 \operatorname{div}((\nabla \psi)(D^\alpha P)) + (D^\alpha P) \Delta \psi \\ &= \psi \partial_{\alpha_1} \partial_{\alpha_2} H - 2 \operatorname{div}((\nabla \psi)(D^\alpha P)) + (D^\alpha P) \Delta \psi \\ &= -\Delta Q_1 - \Delta Q_2 - \Delta Q_3 \end{aligned}$$

where

$$\begin{aligned} -\Delta Q_1 &= \partial_{\alpha_1} \partial_{\alpha_2} (\psi H), \\ -\Delta Q_2 &= -\partial_{\alpha_2} [(\partial_{\alpha_1} \psi)(H)] - \partial_{\alpha_1} [(\partial_{\alpha_2} \psi)(H)] + (\partial_{\alpha_1} \partial_{\alpha_2} \psi)(H) \quad \text{and} \\ -\Delta Q_3 &= -2 \operatorname{div}((\nabla \psi)(D^\alpha P)) + (D^\alpha P) \Delta \psi. \end{aligned}$$

Here  $Q_2$  and  $Q_3$  are defined by the representation formula  $(-\Delta)^{-1}(f) = \frac{1}{4\pi}(\frac{1}{|x|} * f)$

while  $Q_1$  by the Riesz transforms.

Then, by the Riesz transform,

$$\begin{aligned} \|Q_1\|_{L^p(B(b))} &\leq C \|\psi H\|_{L^p(\mathbb{R}^3)} \leq C \|H\|_{L^p(B(a))} \\ &\leq C \|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))}. \end{aligned}$$

Moreover, using Sobolev,

$$\begin{aligned} \|Q_1\|_{L^\infty(B(b))} &\leq C \left( \|Q_1\|_{L^4(B(b))} + \|\nabla Q_1\|_{L^4(B(b))} \right) \\ &\leq C \|H\|_{W^{1,4}(B(a))} \leq C \|H\|_{W^{1,\infty}(B(a))} \\ &\leq C \|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))}. \end{aligned}$$

For  $x \in B(b)$ ,

$$\begin{aligned}
|Q_2(x)| &= \left| \frac{1}{4\pi} \int_{(B(b+\frac{2(a-b)}{3})-B(b+\frac{a-b}{3}))} \frac{1}{|x-y|} \left( \partial_{\alpha_2}[(\partial_{\alpha_1}\psi)(H)](y) \right. \right. \\
&\quad \left. \left. - \partial_{\alpha_1}[(\partial_{\alpha_2}\psi)(H)](y) + (\partial_{\alpha_1}\partial_{\alpha_2}\psi)(H)(y) \right) dy \right| \\
&\leq 2\|\nabla\psi\|_{L^\infty} \cdot \sup_{y \in B(b+\frac{a-b}{3})^C} (|\nabla_y \frac{1}{|x-y|}|) \cdot \|H\|_{L^1(B(a))} \\
&\quad + \|\nabla^2\psi\|_{L^\infty} \cdot \sup_{y \in B(b+\frac{a-b}{3})^C} (|\frac{1}{|x-y|}|) \cdot \|H\|_{L^1(B(a))} \\
&\leq C \cdot \|H\|_{L^1(B(a))}
\end{aligned}$$

because  $|x-y| \geq (a-b)/3$ . Likewise, for  $x \in B(b)$ ,

$$\begin{aligned}
|Q_3(x)| &\leq C \left( \sum_{k=0}^n \|\nabla^{k+1}\psi\|_{L^\infty} \right) \cdot \left( \sum_{k=0}^n \sup_{y \in B(b+\frac{a-b}{3})^C} |\nabla_y^{k+1} \frac{1}{|x-y|}| \right) \cdot \|P\|_{L^1(B(a))} \\
&\quad + C \left( \sum_{k=0}^n \|\nabla^{k+2}\psi\|_{L^\infty} \right) \cdot \left( \sum_{k=0}^n \sup_{y \in B(b+\frac{a-b}{3})^C} |\nabla_y^k \frac{1}{|x-y|}| \right) \cdot \|P\|_{L^1(B(a))} \\
&\leq C \cdot \|P\|_{L^1(B(a))}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\nabla^n P\|_{L^p(B(b))} &\leq \|Q_1\|_{L^p(B(b))} + C\|Q_2\| + \|Q_3\|_{L^\infty(B(b))} \\
&\leq C \cdot \|H\|_{L^p(B(a))} + C \cdot \|H\|_{L^1(B(a))} + C \cdot \|P\|_{L^1(B(a))} \\
&\leq C_{a,b,p,n} \left( \|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))} + \|P\|_{L^1(B(a))} \right)
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla^n P\|_{L^\infty(B(b))} &\leq \|Q_1\| + \|Q_2\| + \|Q_3\|_{L^\infty(B(b))} \\
&\leq C \cdot \|H\|_{W^{1,\infty}(B(a))} + C \cdot \|H\|_{L^1(B(a))} + C \cdot \|P\|_{L^1(B(a))} \\
&\leq C_{a,b,n} \left( \|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))} + \|P\|_{L^1(B(a))} \right).
\end{aligned}$$

□

*proof for lemma 4.3.* We fix  $(n, a, b)$  such that  $n \geq 0$  and  $0 < b < a < 1$  and let  $\alpha$  be a multi index with  $|\alpha| = n$ . Then, by taking  $D^\alpha$  to (18), we have

$$\begin{aligned}
0 = \partial_t(D^\alpha v_1) + \sum_{\beta \leq \alpha, |\beta| > 0} \binom{\alpha}{\beta} ((D^\beta v_2) \cdot \nabla)(D^{\alpha-\beta} v_1) + (v_2 \cdot \nabla)(D^\alpha v_1) \\
+ \nabla(D^\alpha P) - \Delta(D^\alpha v_1).
\end{aligned} \tag{103}$$

We define  $\Phi(t, x) \in C^\infty$  by  $0 \leq \Phi \leq 1$ ,  $\Phi = 1$  on  $Q_b$  and  $\Phi = 0$  on  $Q_a^C$ .

We observe that, for  $p \geq \frac{1}{2}$  and for  $f \in C^\infty$ ,

$$(p + \frac{1}{2})|f|^{p-\frac{3}{2}}f \cdot \partial_x f = \partial_x |f|^{p+\frac{1}{2}} \text{ and } (p + \frac{1}{2})|f|^{p-\frac{3}{2}}f \cdot \Delta f \leq \Delta(|f|^{p+\frac{1}{2}}).$$

which can be verified by direct computations with the fact  $|\nabla f| \geq |\nabla|f||$ .

Now we multiply  $(p + \frac{1}{2})\Phi \frac{D^\alpha v_1}{|D^\alpha v_1|^{(3/2)-p}}$  to (103), use the above observation and integrate in  $x$ . Then we have for any  $p \geq \frac{1}{2}$ ,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \Phi(t, x) |D^\alpha v_1(t, x)|^{p+\frac{1}{2}} dx \\ & \leq \int_{\mathbb{R}^3} (|\partial_t \Phi(t, x)| + |\Delta \Phi(t, x)|) |D^\alpha v_1(t, x)|^{p+\frac{1}{2}} dx \\ & \quad + (p + \frac{1}{2}) \int_{\mathbb{R}^3} |\nabla D^\alpha P(t, x)| |D^\alpha v_1(t, x)|^{p-\frac{1}{2}} dx \\ & \quad + (p + \frac{1}{2}) \sum_{\beta \leq \alpha, |\beta| > 0} \binom{\alpha}{\beta} \int_{\mathbb{R}^3} |(D^\beta v_2(t, x) \cdot \nabla) D^{\alpha-\beta} v_1(t, x)| |D^\alpha v_1(t, x)|^{p-\frac{1}{2}} dx \\ & \quad - \int_{\mathbb{R}^3} \Phi(t, x) (v_2(t, x) \cdot \nabla) (|D^\alpha v_1(t, x)|^{p+\frac{1}{2}}) dx \\ & \leq C \|\nabla^n v_1(t, \cdot)\|^{p+\frac{1}{2}}_{L^1(B(a))} \\ & \quad + C \|\nabla^{n+1} P(t, \cdot)\|_{L^{2p}(B(a))} \cdot \|\nabla^n v_1(t, \cdot)\|^{p-\frac{1}{2}}_{L^{\frac{2p}{2p-1}}(B(a))} \\ & \quad + C \|v_2(t, \cdot)\|_{W^{n,\infty}(B(a))} \cdot \|v_1(t, \cdot)\|_{W^{n,p+\frac{1}{2}}(B(a))} \cdot \|\nabla^n v_1(t, \cdot)\|^{p-\frac{1}{2}}_{L^{\frac{p+\frac{1}{2}}{p-\frac{1}{2}}}(B(a))} \\ & \quad - \int_{\mathbb{R}^3} \Phi(t, x) \operatorname{div} \left( v_2(t, x) \otimes |D^\alpha v_1(t, x)|^{p+\frac{1}{2}} \right) dx \\ & \leq C \|v_1(t, \cdot)\|^{p+\frac{1}{2}}_{W^{n,p+\frac{1}{2}}(B(a))} \\ & \quad + C \|\nabla^{n+1} P(t, \cdot)\|_{L^{2p}(B(a))} \cdot \|\nabla^n v_1(t, \cdot)\|^{p-\frac{1}{2}}_{L^p(B(a))} \\ & \quad + C \|v_2(t, \cdot)\|_{W^{n,\infty}(B(a))} \cdot \|v_1(t, \cdot)\|^{p+\frac{1}{2}}_{W^{n,p+\frac{1}{2}}(B(a))} \\ & \quad + C \|v_2(t, \cdot)\|_{L^\infty(B(a))} \cdot \|\nabla^n v_1(t, \cdot)\|^{p+\frac{1}{2}}_{L^{p+\frac{1}{2}}(B(a))}. \end{aligned}$$



Then integrating on  $[-a^2, t]$  for any  $t \in [-b^2, 0]$  gives

$$\begin{aligned}
& \|D^\alpha v_1\|_{L^\infty(-(b)^2, 0; L^{p+\frac{1}{2}}(B(b)))}^{p+\frac{1}{2}} \\
& \leq C \|v_1\|_{L^{p+\frac{1}{2}}(-(a)^2, 0; W^{n, p+\frac{1}{2}}(B(a)))}^{p+\frac{1}{2}} \\
& \quad + C \|\nabla^{n+1} P\|_{L^1(-(a)^2, 0; L^{2p}(B(a)))} \cdot \|\nabla^n v_1\|_{L^\infty(-(a)^2, 0; L^p(B(a)))}^{p-\frac{1}{2}} \\
& \quad + C \|v_2\|_{L^2(-(a)^2, 0; W^{n, \infty}(B(a)))} \cdot \|v_1\|_{L^{2p+1}(-(a)^2, 0; W^{n, p+\frac{1}{2}}(B(a)))}^{p+\frac{1}{2}} \\
& \quad + C \|v_2\|_{L^2(-(a)^2, 0; L^\infty(B(a)))} \cdot \|\nabla^n v_1\|_{L^{2p+1}(-(a)^2, 0; L^{p+\frac{1}{2}}(B(a)))}^{p+\frac{1}{2}}
\end{aligned}$$

Thus for the case  $p = 1/2$ , we have

$$\begin{aligned}
& \|D^\alpha v_1\|_{L^\infty(-(b)^2, 0; L^1(B(b)))} \\
& \leq C \left[ \left( \|v_2\|_{L^2(-a^2, 0; W^{n, \infty}(B(a)))} + 1 \right) \cdot \|v_1\|_{L^2(-a^2, 0; W^{n, 1}(B(a)))} \right. \\
& \quad \left. + \|\nabla^{n+1} P\|_{L^1(-a^2, 0; L^1(B(a)))} \right]
\end{aligned}$$

while, for the case  $p \geq 1$ , we have

$$\begin{aligned}
& \|D^\alpha v_1\|_{L^\infty(-(b)^2, 0; L^{p+\frac{1}{2}}(B(b)))}^{p+\frac{1}{2}} \\
& \leq C \left( \|v_2\|_{L^2(-a^2, 0; W^{n, \infty}(B(a)))} + 1 \right) \\
& \quad \cdot \left( \|v_1\|_{L^2(-(a)^2, 0; W^{n, 2p}(B(a)))}^{\frac{1}{p+\frac{1}{2}}} \cdot \|v_1\|_{L^\infty(-(a)^2, 0; W^{n, p}(B(a)))}^{1-\frac{1}{p+\frac{1}{2}}} \right)^{p+\frac{1}{2}} \\
& \quad + C \|\nabla^{n+1} P\|_{L^1(-(a)^2, 0; L^{2p}(B(a)))} \cdot \|v_1\|_{L^\infty(-(a)^2, 0; W^{n, p}(B(a)))}^{p-\frac{1}{2}} \\
& \leq C_{a, b, n, p} \left[ \left( \|v_2\|_{L^2(-a^2, 0; W^{n, \infty}(B(a)))} + 1 \right) \cdot \|v_1\|_{L^2(-(a)^2, 0; W^{n, 2p}(B(a)))} \right. \\
& \quad \left. + \|\nabla^{n+1} P\|_{L^1(-(a)^2, 0; L^{2p}(B(a)))} \right] \cdot \|v_1\|_{L^\infty(-(a)^2, 0; W^{n, p}(B(a)))}^{p-\frac{1}{2}}.
\end{aligned}$$

□

*proof for lemma 4.4.* Fix any  $M_0 > 0$  and  $1 \leq p < \infty$  first. Then, for any  $M \geq M_0$  and for any  $f \in C^1(\mathbb{R}^3)$  such that  $\int_{\mathbb{R}^3} \phi(x) f(x) dx = 0$ , we have

$$\begin{aligned}
& \|f\|_{L^p(B(M))} = \left( \int_{B(M)} \left| \int_{\mathbb{R}^3} (f(x) - f(y)) \phi(y) dy \right|^p dx \right)^{1/p} \\
& \leq C \left( \int_{B(M)} \left( \int_{B(1)} |f(x) - f(y)| dy \right)^p dx \right)^{1/p} \\
& \leq C \left( \int_{B(M)} \left( \int_{B(1)} \int_0^1 |(\nabla f)((1-t)x + ty) \cdot (x-y)| dt dy \right)^p dx \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq C(M+1) \left( \int_{B(M)} \left( \int_{B(1)} \int_0^1 |(\nabla f)((1-t)x + ty)|^p dt dy \right)^p dx \right)^{1/p} \\
&\leq C(M+1) \left( \int_{B(M)} \left( \int_{B(1)} \int_0^{\frac{M}{M+1}} |(\nabla f)((1-t)x + ty)|^p dt dy \right)^p dx \right)^{1/p} \\
&\quad + C(M+1) \left( \int_{B(M)} \left( \int_{B(1)} \int_{\frac{M}{M+1}}^1 |(\nabla f)((1-t)x + ty)|^p dt dy \right)^p dx \right)^{1/p} \\
&= (I) + (II)
\end{aligned}$$

where we used  $x \in B(M)$  and  $y \in B(1)$ .

For (I),

$$\begin{aligned}
(I) &\leq C_{M_0} \left( \int_{B(1)} \int_0^{\frac{M}{M+1}} \left( \int_{B(M)} |(\nabla f)((1-t)x + ty)|^p dx \right)^{1/p} dt dy \right) \\
&\leq C_{M_0} \cdot M \int_0^{\frac{M}{M+1}} \frac{1}{(1-t)^{3/p}} \left( \int_{B((1-t)M+1)} |(\nabla f)(z)|^p dz \right)^{1/p} dt \\
&\leq C_{M_0} \cdot M \int_0^{\frac{M}{M+1}} \frac{1}{(1-t)^{3/p}} \left( \int_{B(1)} \int_{B((1-t)M+2)} |(\nabla f)(z+u)|^p dz du \right)^{1/p} dt \\
&\leq C_{M_0} \cdot M \int_0^{\frac{M}{M+1}} \frac{((1-t)M+2)^{3/p}}{(1-t)^{3/p}} \left( \int_{B(1)} \mathcal{M}(|\nabla f|^p)(u) du \right)^{1/p} dt \\
&\leq C_{M_0,p} \cdot M \cdot \|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p} \int_0^{\frac{M}{M+1}} \left( M^{3/p} + \frac{1}{(1-t)^{3/p}} \right) dt \\
&\leq C_{M_0,p} \cdot M \cdot \|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p} \left( M^{3/p} + \int_{\frac{1}{M+1}}^1 \frac{1}{s^{3/p}} ds \right) \\
&\leq C_{M_0} \cdot M \cdot \|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p} \left( M^{3/p} + (M+1)^{3/p} \right) \\
&\leq C_{M_0,p} \cdot M^{1+\frac{3}{p}} \cdot \|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p}
\end{aligned}$$

where we used an integral version of the Minkoski's inequality and  $(1+M) \leq C_{M_0} \cdot M$  from  $M \geq M_0$  for the first inequality.

For (II), observe that if  $\frac{M}{M+1} \leq t \leq 1$ , then  $0 \leq 1-t \leq \frac{1}{M+1}$  and

$$|(1-t)x + ty| \leq (1-t) \cdot |x| + t|y| \leq \frac{M}{M+1} + 1 \leq 2$$

because  $x \in B(M)$  and  $y \in B(1)$ . Thus

$$\begin{aligned}
 (II) &\leq C_{M_0} \cdot M \left( \int_{B(M)} \left( \int_{\frac{M}{M+1}}^1 \frac{1}{t^3} \int_{B(2)} |(\nabla f)(z)| dz dt \right)^p dx \right)^{1/p} \\
 &\leq C_{M_0} \cdot M \cdot M^{3/p} \cdot \int_{B(2)} |(\nabla f)(z)| dz \cdot \int_{\frac{M}{M+1}}^1 \frac{1}{t^3} dt \\
 &\leq C_{M_0} M^{1+\frac{3}{p}} \cdot \|\nabla f\|_{L^1(B(2))}.
 \end{aligned}$$

□

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